INSTITUTE
OF
HYDROLOGY

DERIVATION OF A CATCHMENT AVERAGE UNIT HYDROGRAPH

by

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ABSTRACT

A problem that frequently arises in rainfall/runoff modelling is the derivation of a catchment average unit hydrograph from a number of recorded flood events. Two approaches are considered in this report: averaging unit hydrographs derived from individual events and joint analysis of a group of events to determine an average unit hydrograph directly. The problems of instability which often feature in least-squares unit hydrographs derived from individual events are found to have little influence on average unit hydrographs determined for eight trial catchments. Whilst particular techniques have their merits, the main conclusion is that differences are unlikely to be significant in application. Of the methods considered, event superposition followed by a least-squares solution is favoured for computational economy. Replacing the least-squares criterion by an iterative solution of the dominant equations is shown to bring derivation of an average unit hydrograph within the scope of hand calculation.

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CONTENTS

1 INTRODUCTION

1.1 Concept of a catchment average unit hydrograph

Methods of rainfall runoff modelling based on unit hydrograph theory continue to find application in design flood estimation and flood forecasting. Much of the current interest in unit hydrographs stems from their use in the Flood Studies Report rainfall/runoff method for flood estimation on ungauged catchments (FSR I.6*). A unit hydrograph technique has perhaps two important advantages over statistical methods such as flood formulae based on multiple regression. Firstly, it allows estimation of the whole hydrograph rather than just a single feature (eg peak flow). A second advantage lies in the flexibility that unit hydrograph methods provide to make good use of local rainfall and runoff data. If several flood events are available for analysis on a given catchment then the usual requirement is to determine a single unit hydrograph that characterizes the catchment response to heavy rainfall. Such an entity can be termed a catchment average unit hydrograph.

Of course the adoption of a fixed response function for a catchment has limitations. The generation of runoff from rainfall is ffected by a host of physical processes and spatial factors and, irrespective of the definitions used, the assumption of a linear time-invariant relationship between net rainfall and response runoff is no more than an approximation. However, such evidence as is available (see FSR I.6.5.3) suggests that systematic variation of unit hydrographs between events is discernible on relatively few catchments. Moreover, where present, the systematic variation is not always in accordance with the intuitive reasoning that bigger floods propagate more rapidly.

When appraising the significance of variations between unit hydrographs it is important to appreciate that calibration of the rainfall/runoff model involves more than derivation of a suitable unit hydrograph; rules must be formulated for estimating net rainfall and baseflow (see Figure 1.1). Rainfall separation is a

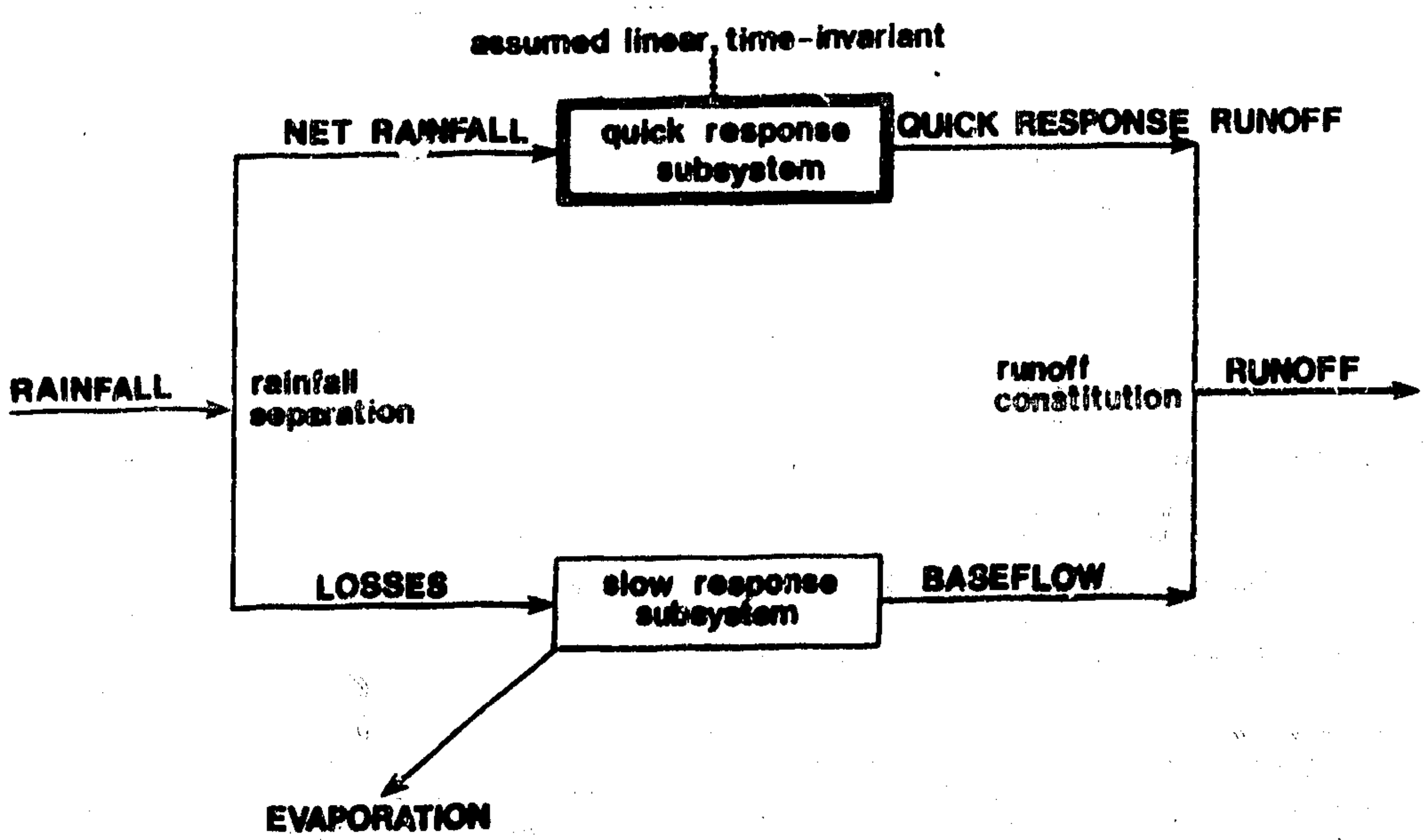


FIGURE 1.1 A system representation of the unit hydrograph approach

crucial consideration because it determines the volume of response runoff. In contrast the unit hydrograph fulfils what is perhaps a secondary role, namely distributing this volume in time. Thus, while studies of differences between unit hydrographs may be interesting and occasionally informative, a sufficient requirement in many cases is for a catchment average unit hydrograph. The report offers guidance in this one aspect of rainfall/runoff modelling, namely derivation of an average unit hydrograph from a number of recorded events on a catchment.

1.2 Data selection and separation

The study took flood event data for eight catchments from the archive assembled at the Institute of Hydrology for the UK Flood Studies and follow-up work. The decision to study eight catchments was fairly arbitrary. Too small a number would have provided little opportunity to draw generalizations, whilst many more than eight would have made the study unwieldy. The selection of catchments took many factors into account, albeit informally. The criteria for acceptance included the requirements that the catchment should: (i) have at least ten events available for analysis, (ii) exhibit a significant quick response to heavy rainfall, and (iii) be amenable to analysis at 1 hour data interval. Within these limits an attempt was made to arrive at a set of eight catchments that embraced as wide a range of catchment sizes as possible.

The outcome of the procedure was to select the catchments listed in Table 1.1. As it transpired all are upland catchments, the smallest having an area of 69 km^2, the largest one of 298 km^2. Included in the table are some of the more interesting catchment characteristics (definitions as per FSR I.4.2).

Net rainfall and quick response runoff data were determined by the procedures used in the flood studies project. Separation of runoff is by first extrapolating the antecedent recession and then drawing a straight line from the time of peak runoff to rejoin the hydrograph at a predetermined time after the cessation of rainfall. This time is taken as four times the lag between the centroid of the hyetograph and centroid of peaks (if more than one) of the hydrograph. The method of rainfall separation can be summarised as a variable loss rate indexed by catchment wetness, the free parameter being determined by the condition that the volumes of net rainfall and quick response runoff should be equal. A full description of these methods of data separation is given in FSR I.6.4. While alternative separation procedures would undoubtedly have resulted in different unit hydrographs being derived, the choice was considered unimportant in the context of comparing methods of averaging.

1.3 Structure of report

The report examines the features and relative merits of a number of methods of determining an average unit hydrograph from several net rainfall/quick response runoff events. Two distinct approaches are considered: averaging unit hydrographs derived from the individual events (Section 3) and joint analysis of the group of events to determine an average unit hydrograph directly (Section 4). A necessary ingredient for either approach is a method of unit hydrograph derivation (Section 2). The comparative element of the report culminates in Section 5, which paves the way for discussion of a particularly simple method of determining a catchment average unit hydrograph (Section 6). The final section summarises the conclusions reached and makes recommendations for further study.

**References* to the *Flood Studies Report* (Natural Environment Research Council (1975)) are by volume numeral and chapter, section or subsection number.*

TABLE 1.1 DETAILS OF CATCHMENTS SELECTED FOR STUDY

Catchment		Number of Events	Catchment characteristics						
Number	Name		AREA km^2	MSL km	S1085 m/km	SOIL	RSMD mm	URBAN	LAKE
23002	Derwent at Eddys Bridge	10	118	22.8	10.7	0.500	42.3	-	-
45004	Axe at Whitford	11	298	34.5	3.8	0.279	42.9	.006	-
46003	Dart at Austins Bridge	22	248	35.2	6.5	0.361	71.9	-	.013
53005	Midford Brook at Midford	14	147	24.6	3.0	0.247	36.6	.009	-
58001	Ogmore at Bridgend	17	158	20.2	10.3	0.469	56.6	.036	-
61001	Western Cleddau at Prendergast Mill	21	198	26.4	2.7	0.329	45.5	-	-
65001	Glaslyn at Beddgelert	15	69	15.3	33.4	0.500	109.1	-	-
76014	Eden at Kirkby Stephen	15	69	20.1	19.7	0.460	52.2	-	-

2 METHODS OF UNIT HYDROGRAPH DERIVATION

2.1 Introduction

Only one basic class of methods of unit hydrograph derivation is considered in this report. These are ordinate methods in which the unit hydrograph is represented by a sequence of ordinates evenly spaced in time. The convolution relation is re-formulated as a set of simultaneous linear equations which is solved to determine the unit hydrograph ordinates from net rainfall and quick response runoff data.

There are many other methods of unit hydrograph derivation and a comprehensive review is beyond the scope of this report. O'Donnell (1966) distinguishes two approaches to unit hydrograph derivation: techniques of synthesis and techniques of analysis. In many applications it is necessary or desirable to represent the unit hydrograph (or response function) in terms of a small number of parameters. Synthesis techniques fulfil this requirement by assuming a particular parametric form for the response function, a well-known example being the cascading linear reservoir model developed by Nash (1960). However, there is the inherent difficulty that the chosen parametric form may be an inappropriate representation of the characteristic response of a particular catchment.

The analysis approach avoids this problem by invoking general techniques of linear systems analysis to 'invert' the convolution relation:

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau \qquad 2.1$$

to determine the system response function, h(t), from knowledge of the input, x(t), and output, y(t). In the context of rainfall/runoff modelling, x(t) is the net rainfall hyetograph, y(t) the quick response runoff hydrograph, and h(t) the instantaneous unit hydrograph.

If the net rainfall data are taken in block form, and the response runoff data in ordinate form, then the convolution relation simplifies to a number of summations:

$$y_u = \sum_{i=1}^{j} x_i u_{j-i+1} \quad \text{for } j = 1, 2, \ldots \qquad 2.2$$

where $y_j \equiv y(j\Delta T)$, $x_i \equiv x(\tau)$ for $(i-1)\Delta T < \tau \le i\Delta T$, and $U_k \equiv \Delta T.U(\Delta T,k\Delta T)$, $U(\Delta T,t)$ being the ΔT-period unit hydrograph. With appropriate limits on the summations defined during rainfall and runoff separation, Equations 2.2 can be rewritten in matrix form:

$$\begin{vmatrix} x_1 & & \\ x_2 & x_1 & \\ \cdot & x_2 & \cdot \\ \cdot & \cdot & x_1 \\ x_N & \cdot & x_2 \\ & x_N & \cdot \\ & & \cdot \\ & & x_N \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{vmatrix} = \begin{vmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{vmatrix} \qquad 2.3$$

or

$$X\underline{u} = \underline{y} \qquad 2.4$$

where X is an m x n coefficient matrix constructed out of the N periods of net rainfall, $\underline{u}$ is the n x 1 vector of unit hydrograph ordinates, and $\underline{y}$ is the m x 1 vector of quick response runoff ordinates. It is usual to determine the length of the unit hydrograph vector by the condition:

$$n = m - N + 1 \qquad 2.5$$

so that the duration, or time base, of the unit hydrograph is consistent with the time bases of the net rainfall hyetograph and quick response runoff hydrograph.

The above definitions are summarised in Figure 2.1. We will refer to Equation 2.4 as the discrete convolution relation; it comprises a system of m simultaneous linear algebraic equations in n unknowns, the unknowns being the unit hydrograph ordinates, u_k. This formulation reduces the problem of unit hydrograph derivation to one of numerical algebra, namely solution of Equations 2.3.

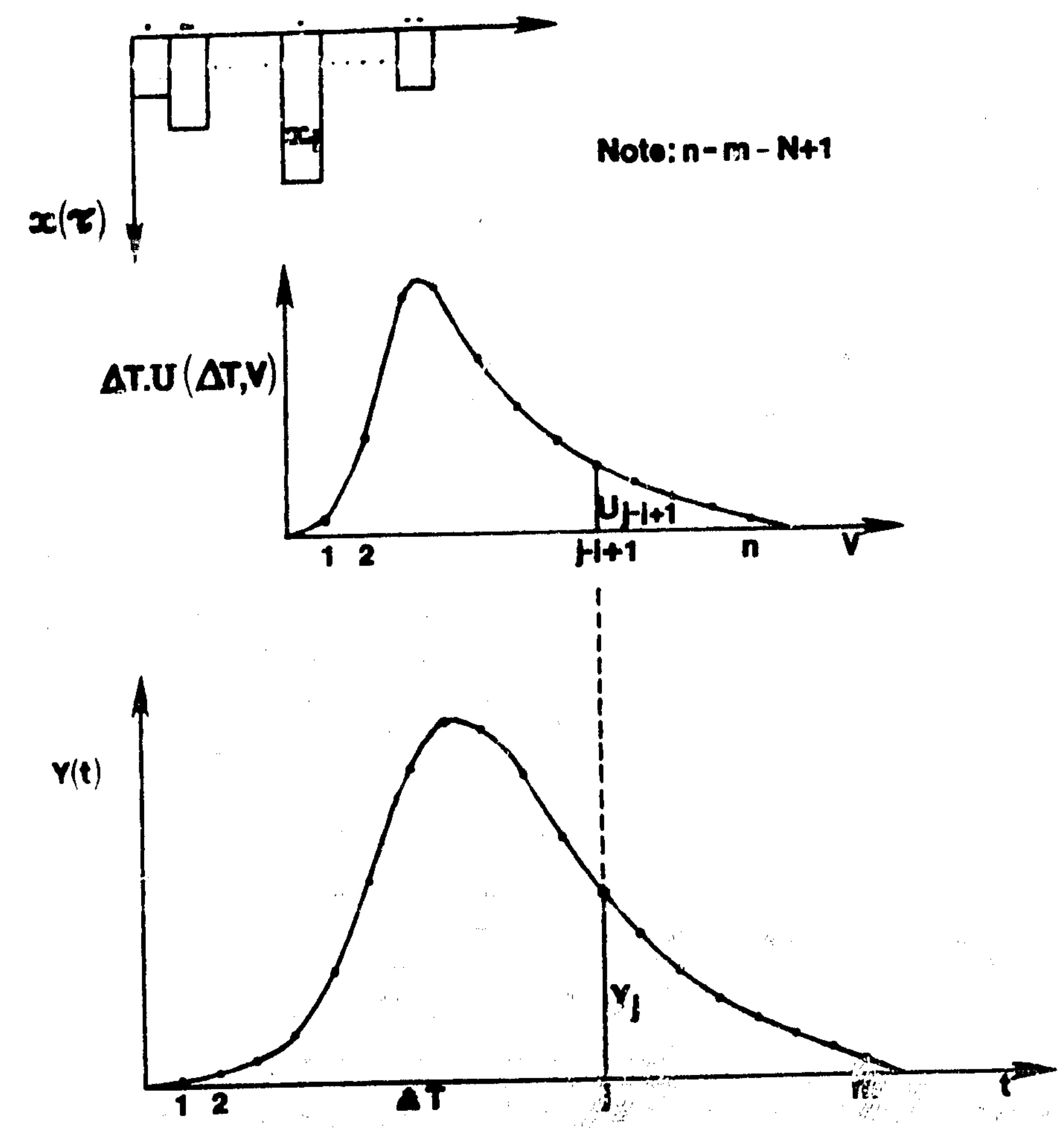


FIGURE 2.1 Discrete formulation of the convolution relation

2.2 Least-squares method

In general the problem posed by the discrete convolution relation is over-determined there being more equations than unknowns ($m > n$). A unique solution can be obtained by applying a least-squares criterion, ie minimization of the residual sum of squares. This is a widely used expedient and a variety of treatments of least-squares methods can be found in numerical algebra and statistical texts (eg Wilkinson and Reinsch (1971), Hoel (1962)).

The usual way of deriving the least-squares solution, $\hat{\underline{u}}$, to Equation 2.4 is to construct the 'normal equations':

$$X^T X \hat{\underline{u}} = X^T \underline{y} \qquad 2.6$$

which can then be solved using a standard technique such as the Choleski symmetrical decomposition. The least-squares solution is nominally written:

$$\hat{\underline{u}} = (X^T X)^{-1} X^T \underline{y} \qquad 2.7$$

although in practice $\hat{\underline{u}}$ is usually obtained without explicitly forming the inverse matrix, $(X^T X)^{-1}$. That this matrix manipulation provides the least-squares estimate of $\underline{u}$ is demonstrated in FSR I.6.4.6 by a simple example. We will refer to solution of the discrete convolution relation using a least-squares criterion as the least-squares ordinate method, or simply the least-squares method, of unit hydrograph derivation.

Application of the least squares method to derive a unit hydrograph from an observed event often produces an unrealistic solution. A common fault is that the unit hydrograph is oscillatory, perhaps containing negative ordinates (eg Figure 2.2). Intuition tells us that the response of a catchment to a pulse of net

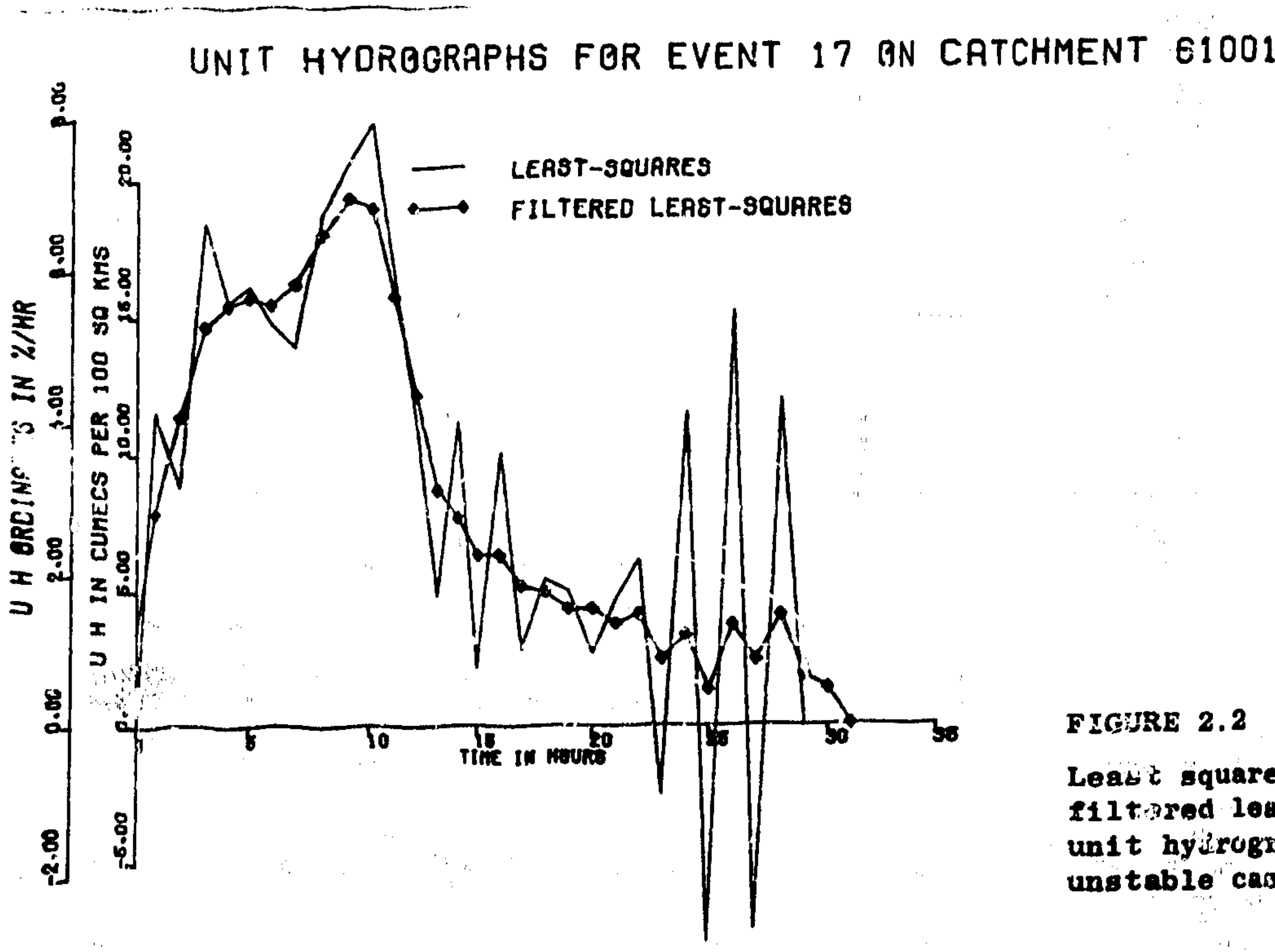


FIGURE 2.2

Least squares and filtered least-squares unit hydrographs - unstable case rainfall ought to be smooth with (usually) just one maximum point and two inflexion points (one on the rising limb and one on the receding limb) - as shown, for example, in Figure 2.3. The anomaly stems from the fact that the unit hydrograph is defined in terms of a large number of parameters (ordinates) and consequently has the flexibility to approximate very closely the observed relationship between net rainfall and quick response runoff. The least squares criterion ensures that it does so, and, in the absence of any restrictions on the relative values of the ordinates, the end-product can be an unstable unit hydrograph.

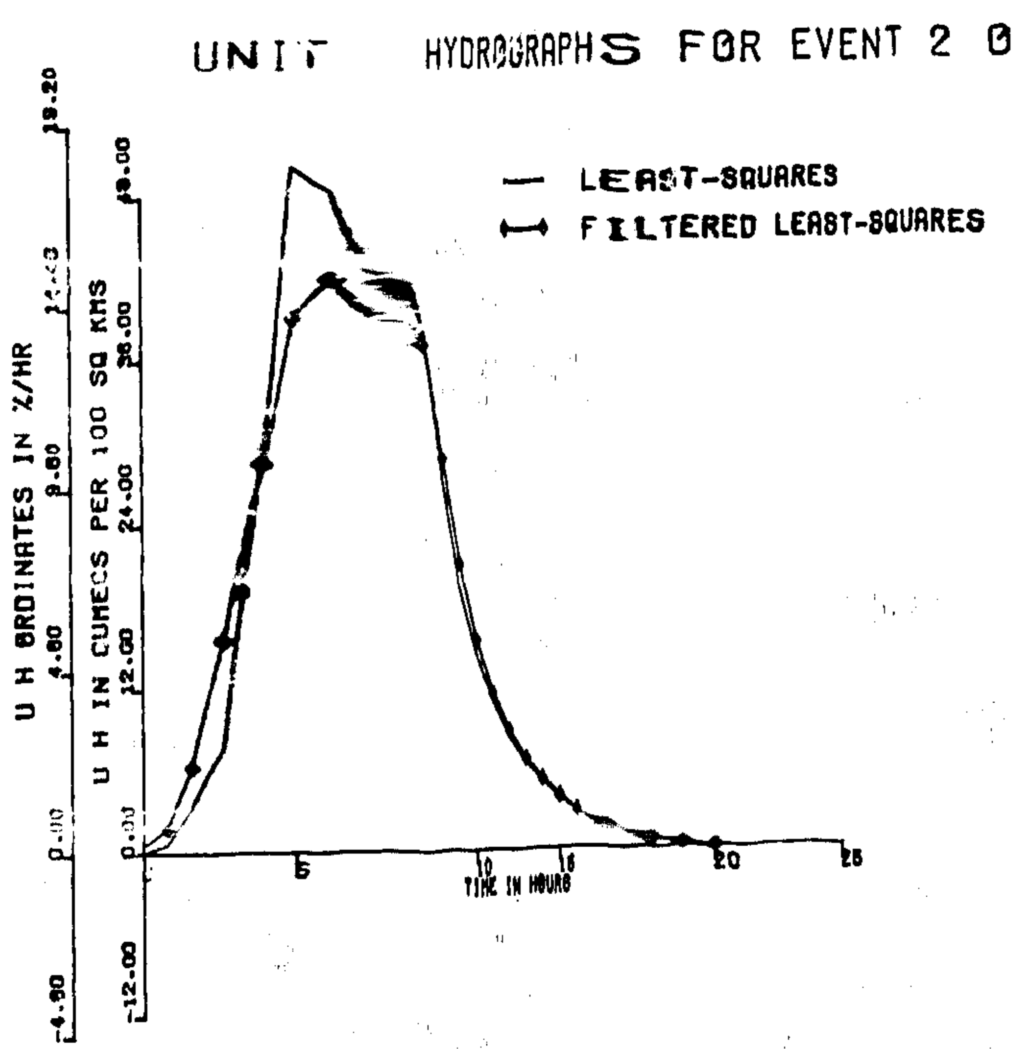


FIGURE 2.3

Least squares and filtered least-squares unit hydrographs - stable case

Some analysts, believing oscillatory unit hydrographs to be, at least in part, due to numerical instabilities of the solution method, reject the commonly used normal equations approach in favour of the Householder method of orthogonal transforms (see, for example, Wilkinson (1965)). The latter method provides a direct least-squares inversion of Equation 2.4 (ie without forming the normal equations) but is a more complicated and computationally demanding technique. While too simple a numerical solution technique (such as forward substitution in Equations 2.3) can induce instabilities in derived unit hydrographs, the chief source of oscillatory solutions lies in the problem itself rather than in the choice of a particular solution method. In the authors' opinion the refinement provided by Householder's method is of little consequence in unit hydrograph derivation problems. For historical reasons, however, two of the methods of unit hydrograph derivation tested in this study use Householder's technique whereas the third (the Flood Studies Report method) uses the normal equations approach.

Various modifications to the least-squares method have been proposed to ensure the stability of derived unit hydrographs. The restricted least-squares method (discussed in Section 2.4) attains a stable result by reducing the number of ordinates defining the unit hydrograph. A simpler alternative is to smooth the derived unit hydrograph; the variant of the least-squares method used in the Flood Studies Report incorporates an element of such post-derivation smoothing, and is considered next.

2.3 Flood Studies Report method

As part of the UK Flood Studies project, unit hydrographs were derived for more than 1400 events using a modified least-squares method. This is referred to in the Flood Studies Report as 'matrix inversion with smoothing'. The smoothing is imparted in two ways.

In the basic least-squares method, m equations are solved in n unknowns, namely Equations 2.3. The first way in which the Flood Studies Report method differs is that m + 16 equations are solved in n + 16 unknowns. This is because the definitions of the unit hydrograph and quick response runoff hydrograph are extended to include an additional 16 ordinates (six before and ten after), the additional quick response runoff ordinates being set to zero. The extended representation is illustrated in Figure 2.4. The effect of this modification has not been fully isolated in the present study but it is thought that the extra values help to dampen instabilities, particularly those occurring towards the start and end of the unit hydrograph. Certainly the technique allows a degree of flexibility in interpreting rainfall and runoff records that may be poorly synchronized.

The second way in which the Flood Studies Report method differs is that the unit hydrograph is subjected to post-derivation smoothing. The unit hydrograph is passed twice through a three point moving average filter in which each ordinate is replaced by the average of itself and its two neighbours. The filtering process succeeds in damping random fluctuations in the unit hydrograph ordinates but at the expense of reducing the magnitude of true variations. These good and bad features of the moving average filter are amply demonstrated in Figures 2.2 and 2.3 respectively. The extent of either effect is very much dependent on the interval at which the data are sampled.*

One other aspect in which the Flood Studies Report method differs from the basic least-squares method is that the derived and smoothed unit hydrograph is finally scaled to unit volume (ordinates corresponding to negative times being discounted at this stage - See Figure 2.4). There are two schools of thought on this practice. One holds that it is known *a priori* that the response function should have unit volume. The other view is that if the volume of a derived unit hydrograph deviates significantly from unity then it is a signal that all is not well (either with the basic data, the assumptions made, or the methods used) and to scale blindly would be dangerous. In the present study the Flood Studies Report method has been followed in its entirety but it should be noted that in the other methods considered no adjustment for unit volume has been made.

2.4 Restricted least-squares method

The restricted least-squares method was developed by Reed (1976) with the specific aim of overcoming instabilities in derived unit hydrographs. In this method the unit hydrograph is defined in terms of a reduced number of ordinates, no longer evenly spaced in time. Definition of intervening ordinates of the unit hydrograph

**The tendency of the filter to reduce the peaks of unit hydrographs was investigated in the original Floods Studies. As a consequence, the synthetic unit hydrograph used in the flood estimation procedure was adjusted to compensate for this effect (see FSR I.6.4.6 and 6.5.9).*

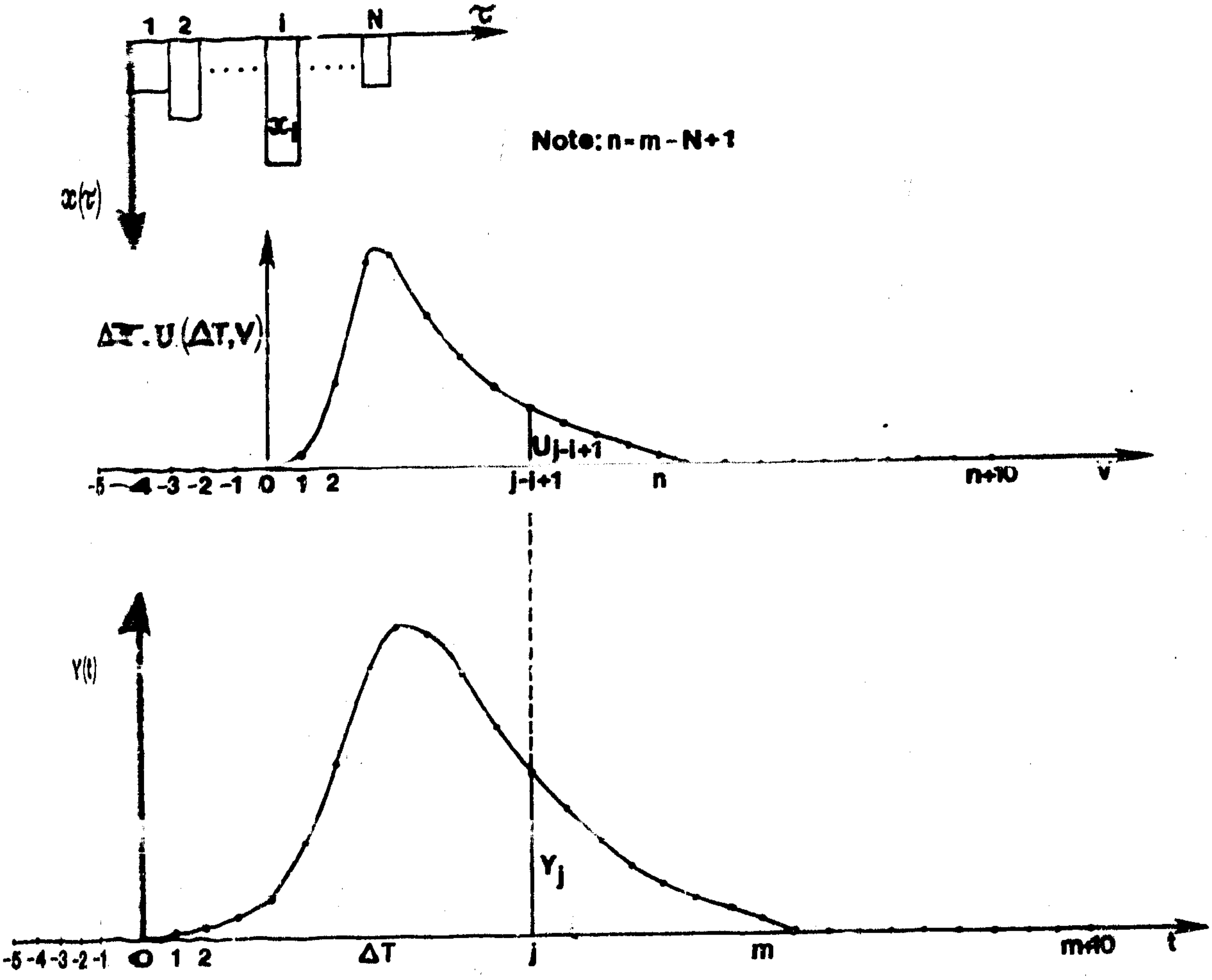


FIGURE 2.4 **Extended representation of the convolution relation (FSR method)**

is by linear interpolation. Starting with the unrestricted unit hydrograph (ie that derived by the basic least-squares method) the restricted least-squares algorithm systematically eliminates ordinates with the effect that the unit hydrograph ultimately derived is unimodal and has the equivalent of no more than one inflexion point on either the rising or receding limb. Details of the method are reproduced in Appendix 1. Examples of unit hydrographs derived by the restricted least-squares method are shown in Figures 2.5 and 2.6.

Of course there may be some catchments where the characteristic response is bimodal, for example as a function of catchment shape or land use. As a precaution against dogmatic use of an inappropriate method, it is wise to examine the unrestricted unit hydrograph for evidence of systematic bimodularity, although usually such behaviour would be evident from a preliminary inspection of the flood event data. For the eight catchments considered here, the general effect of the restricted method was to remove spurious variations from the unrestricted unit hydrograph rather than to impose on the natural shape of the catchment response.

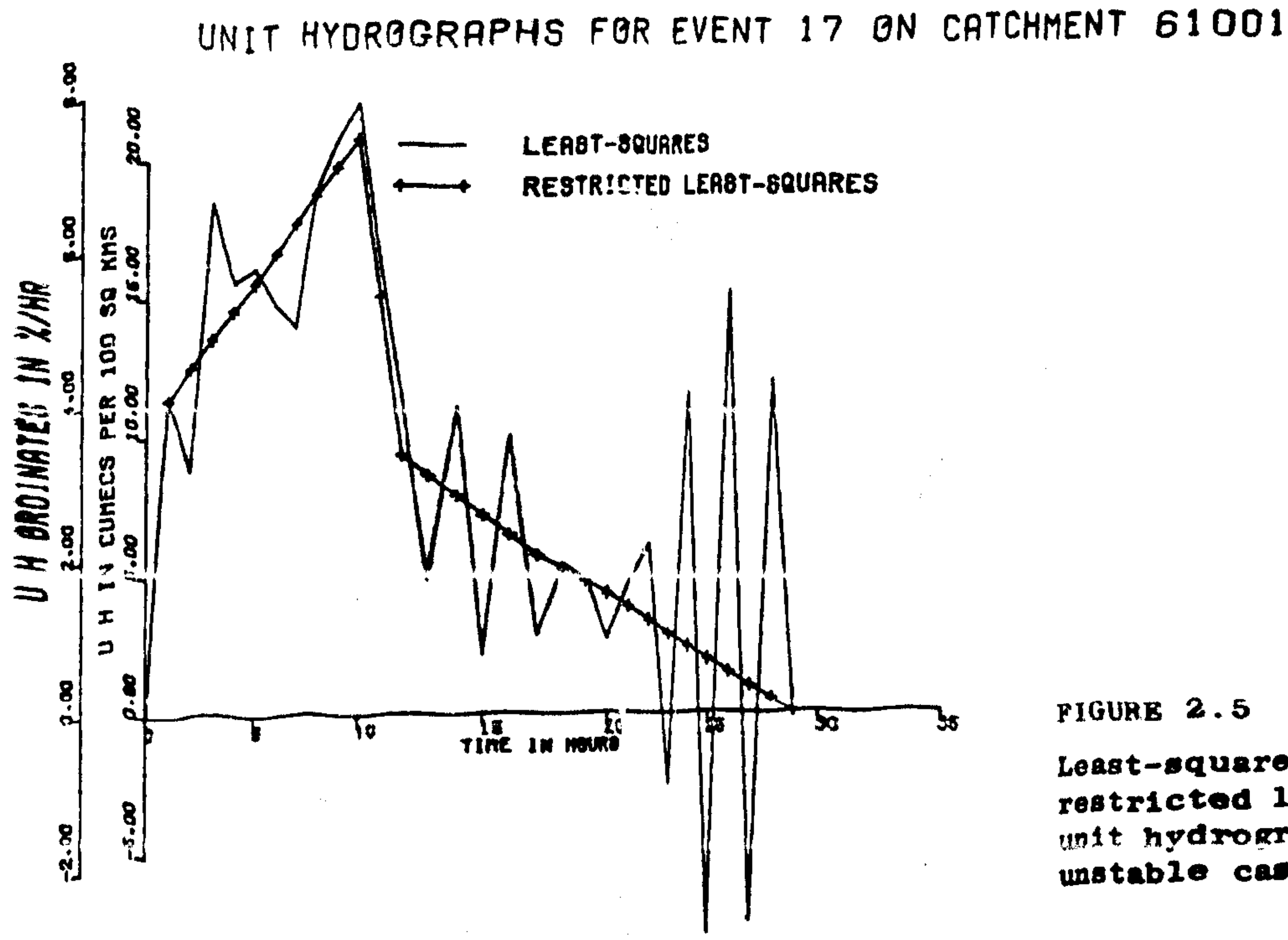


FIGURE 2.5

Least-squares and restricted least-squares unit hydrographs - unstable case

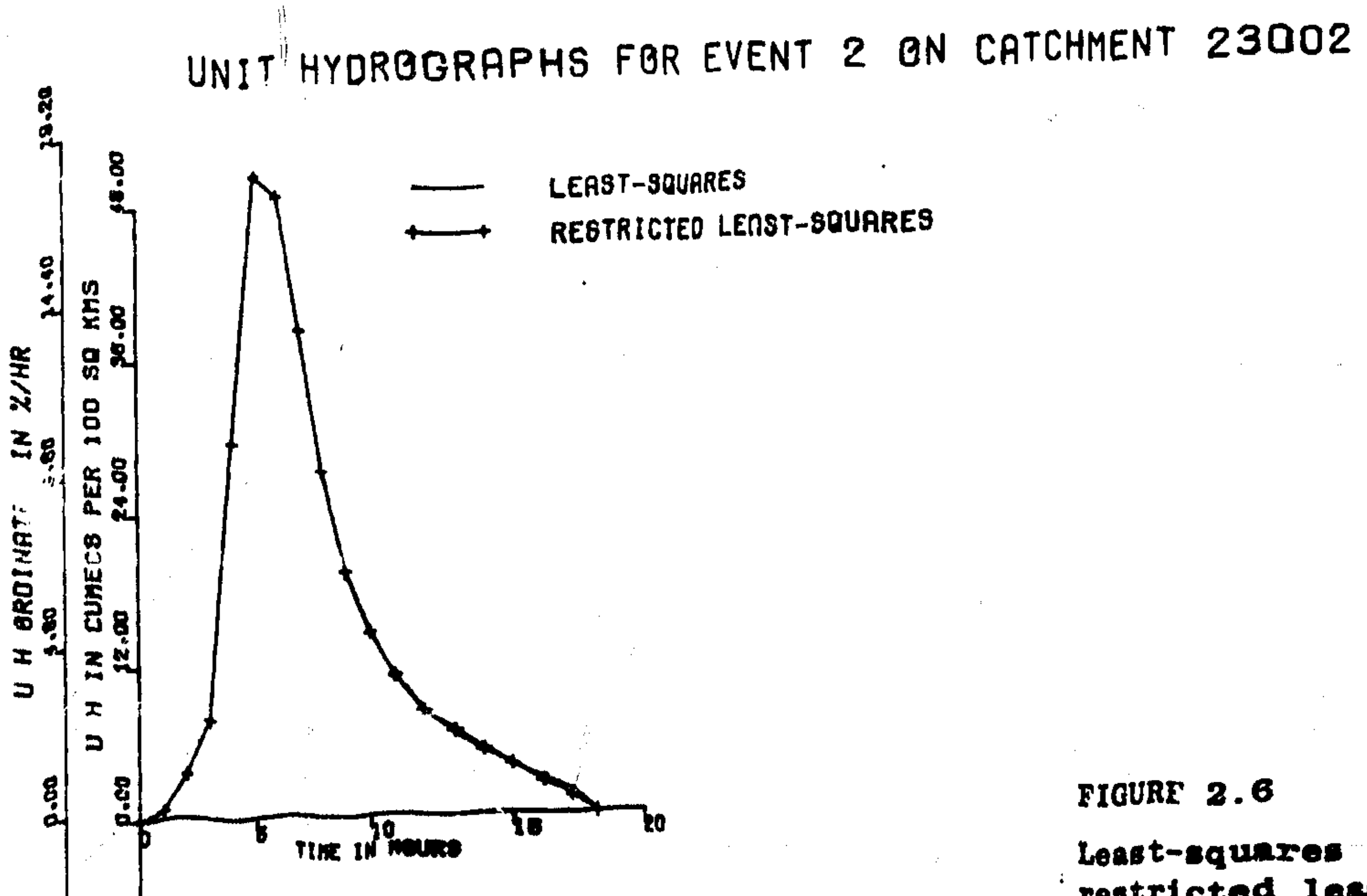


FIGURE 2.6

Least-squares and restricted least-squares unit hydrographs - stable case

2.5 Comparisons

How well the above methods perform with regard to deriving a unit hydrograph from an individual event is peripheral to this study. Rather, it is their effectiveness as part of a technique for deriving an average unit hydrograph that is under scrutiny. However, the example of application to an individual event does serve to illustrate some characteristics of the methods.

Unit hydrographs derived by the three methods under consideration are compared in Figure 2.7 for a particular event. Event 18 on catchment 46003 was selected for illustration because it exemplifies a situation where the least-squares unit hydrograph is in need of some form of smoothing. This was not the least stable of the 125 unit hydrographs derived from individual events by the least-squares method, but it was singled out because it came from a relatively simple event. As Figure 2.8 (overleaf) confirms, the rainfall hyetograph exhibits no unusual features.

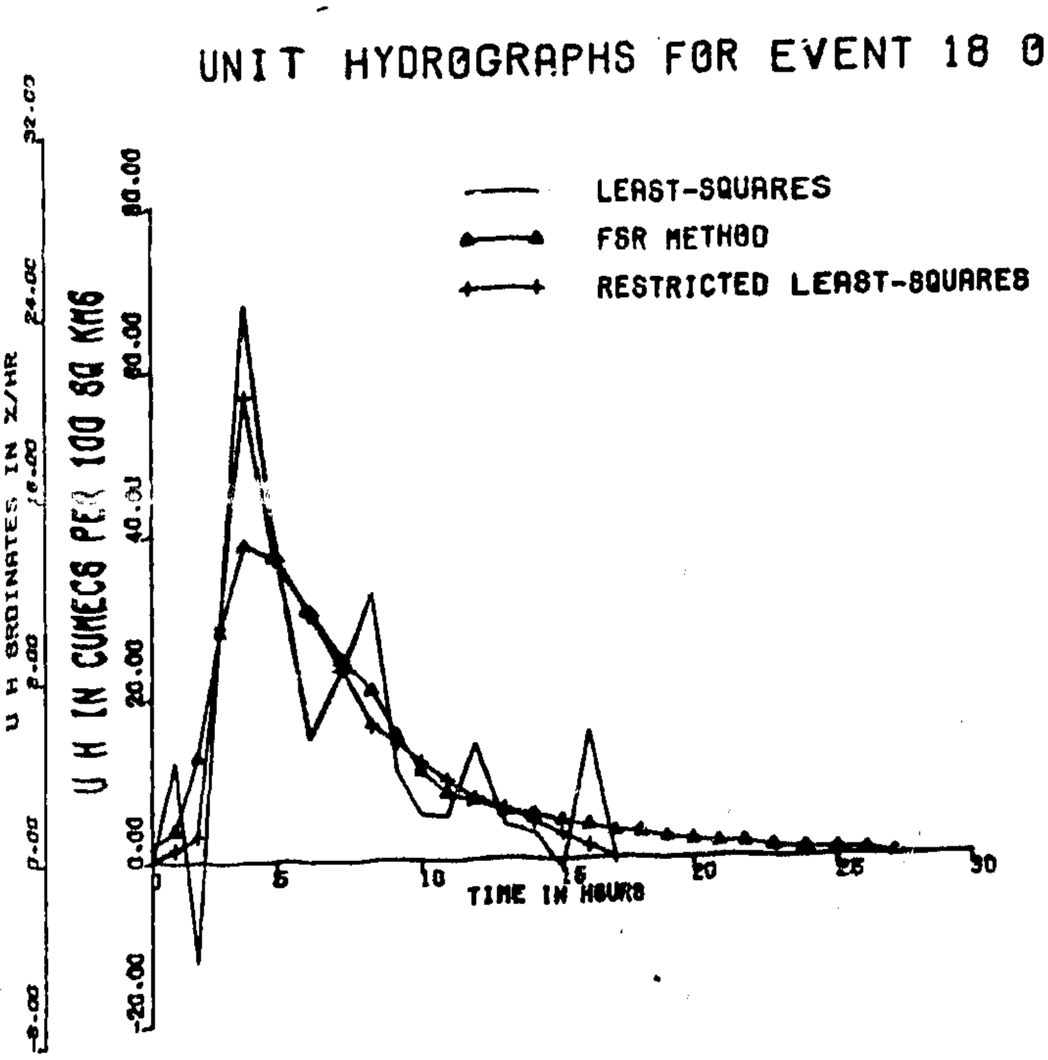


FIGURE 2.7 Comparison of derivation methods

Returning to Figure 2.7, it is seen that in this instance the Flood Studies Report method provides sufficient smoothing to eliminate the oscillations that perturb the least-squares unit hydrograph. However, the tendency of the moving average filter to reduce the magnitude of true variations between ordinates is evident in the less steep rising limb and the lower peak ordinate. The effect of the extended definition used in the Flood Studies Report method is also apparent; note that the initial ordinate is greater than zero and that the receding limb is prolonged. For this event, the adjustment to unit volume required no more than a 1% increase in the ordinates.

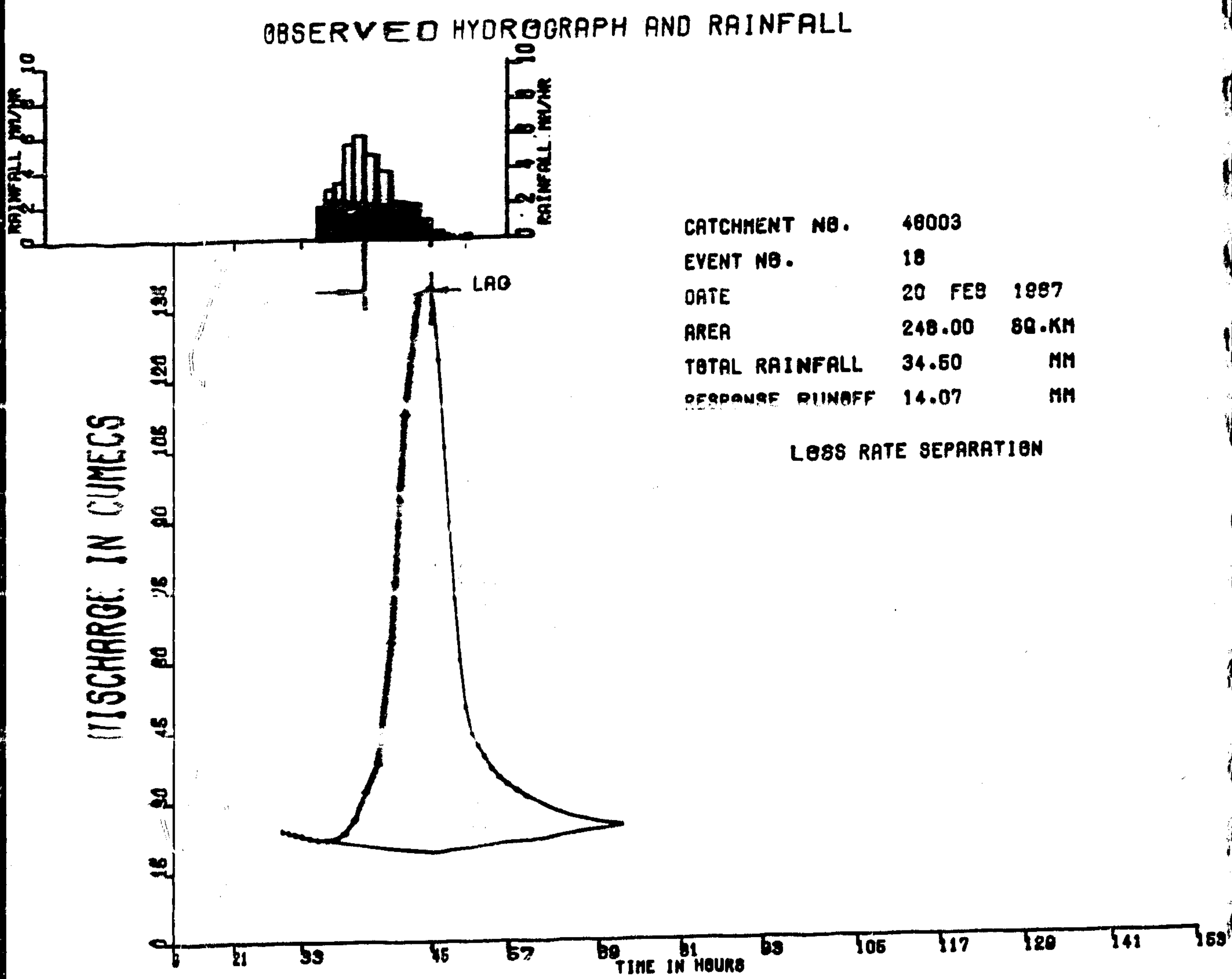


FIGURE 2.8 Data and separation for event 18 on catchment 46003

The unit hydrograph derived by the restricted least-squares method comprises seven line segments; it approximates to what might be obtained if the least-squares unit hydrograph were smoothed by eye, although some of the gradient changes are rather abrupt. While it has to be admitted that the restricted least-squares technique is complicated, and therefore difficult to implement, given the availability of a standard program it would seem to provide a more effective antidote to oscillatory unit hydrographs than more conventional methods of smoothing.

3 AVERAGING TECHNIQUES

3.1 Introduction

There are several ways of constructing or calculating an average unit hydrograph from a number of individually derived unit hydrographs. The textbook method (eg Wilson (1974), Linsley *et al.* (1975)) is to plot the unit hydrographs on a single diagram and to mark on a point which corresponds to the average peak ordinate and average time to peak (see Figure 3.1). An average unit hydrograph is then sketched in by eye so that it passes through the average peak point, has unit volume, and generally conforms to the characteristic shape of the individual unit hydrographs. The procedure relies on a limited amount of trial and error (to preserve volume) and on a good deal of subjective judgement. The ability of the eye to discriminate outliers, and to detect and interpret shape, is a strength that objective averaging procedures find hard to emulate.

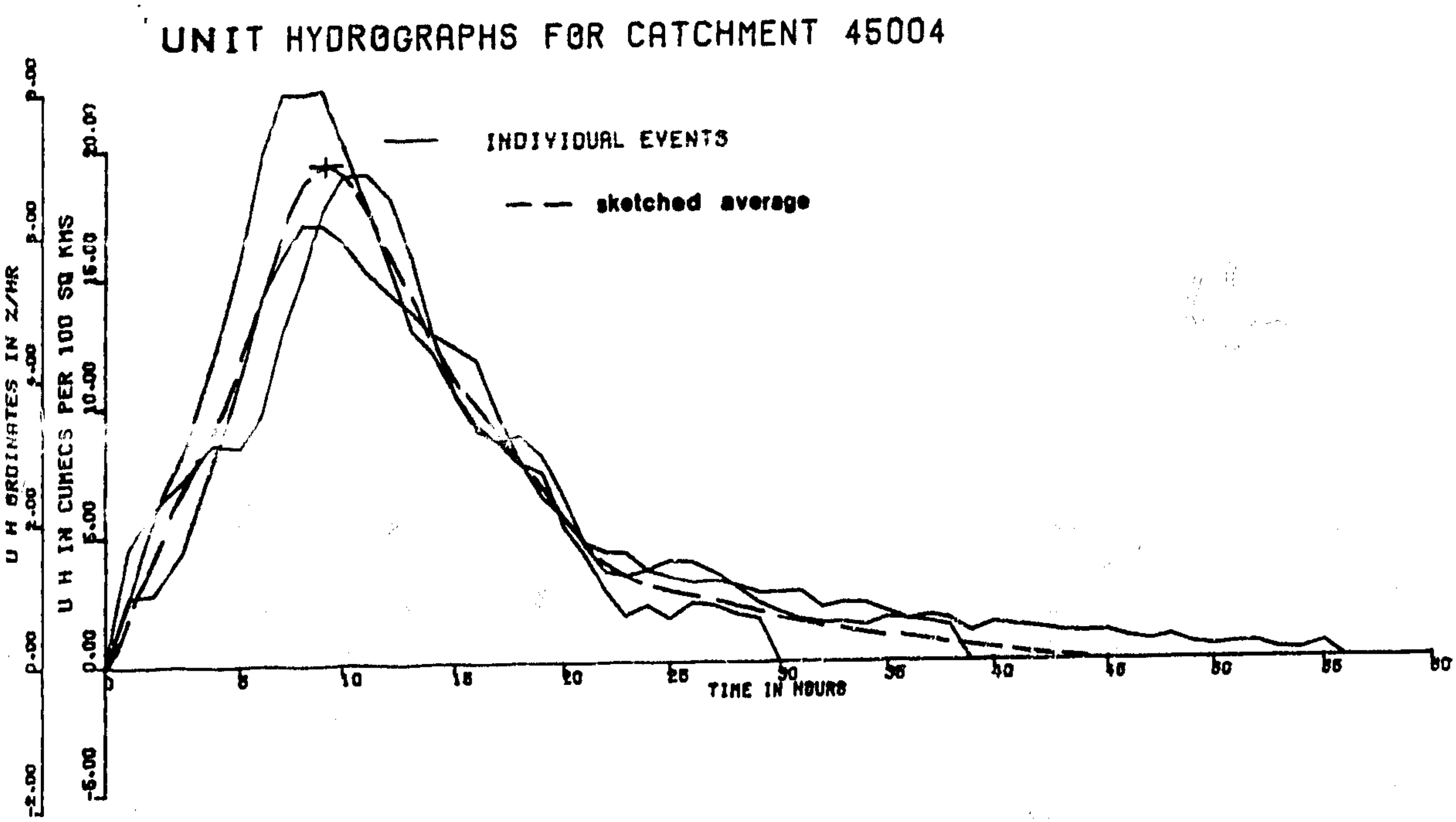


FIGURE 3.1 Textbook method of averaging

The methods of averaging assessed in this study are objective ones, ordinate by ordinate averaging (Section 3.2) and shape factor averaging (Section 3.3). However, comparisons made in Section 3.4 do make reference to 'by eye' results obtained at an early stage of the study.

3.2 Ordinate by ordinate averaging

Various ordinate by ordinate averaging methods are possible and four such schemes were considered. The simplest of these was to construct each ordinate of the average unit hydrograph by taking the mean of the corresponding ordinates of the individual unit hydrographs; this will be referred to as the mean method. A variation of this approach is the median method; use of a median form of averaging reduces the effect of outlandish ordinates.

A refinement of simple ordinate by ordinate averaging is to align the peak ordinates of the individual unit hydrographs prior to averaging. This is the approach recommended in the Flood Studies Report (FSR I.6.8.4) and is more in keeping with the 'by eye' textbook method referred to in Section 3.1. The precise procedure is to calculate the average time to peak of the individual unit hydrographs and then to align the unit hydrographs with their peak ordinates synchronized at this average time. Depending on the form of averaging used, either the mean peaks aligned or median peaks aligned method results. The simple and 'peaks aligned' approaches are contrasted in Figures 3.2a and b.

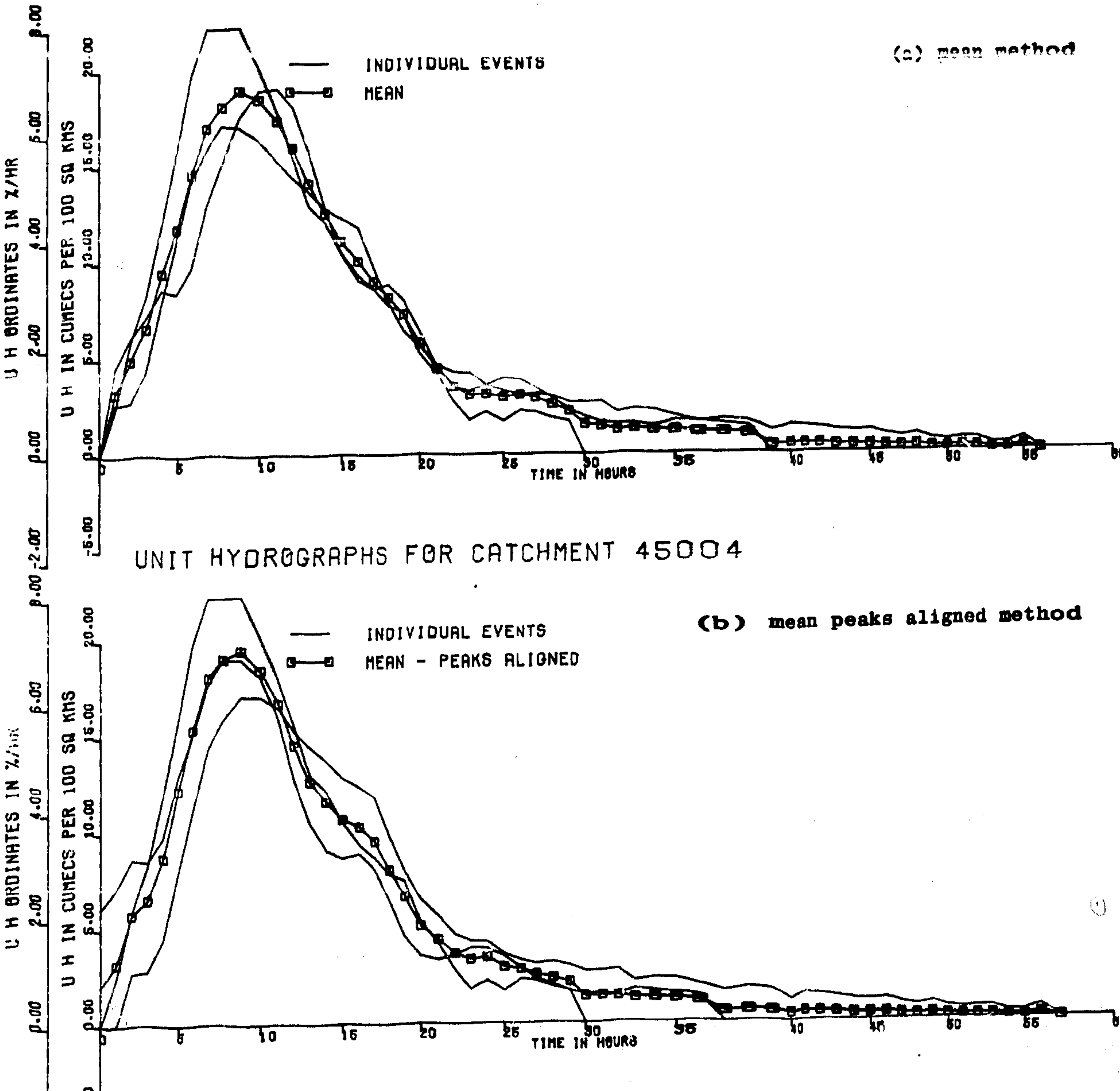


FIGURE 3.2 Ordinate by ordinate averaging

3.3 Shape factor averaging

Objective averaging procedures are hard-pressed to match the eye's ability to discriminate outliers and interpret shape. Use of a median form goes some way to eliminate the effect of outliers but ordinate methods are quite incapable of appraising shape. An alternative approach giving emphasis to shape considerations is the shape factor method developed by Reed (1976).

The basis of the approach is the characterization of unit hydrographs by statistics defined in terms of moment integrals: namely the volume, the mean, and the coefficients of variation, skewness, and peakedness (see Appendix 2 for definitions). Strictly speaking, the volume and mean characterize the scale and location of the distribution (namely the unit hydrograph) rather than its shape, but it is convenient to refer to the five statistics collectively as shape factors.

Reed's method of using shape factors to determine an average unit hydrograph is as follows. First, the shape factor values are calculated for each individual derived unit hydrograph and the values averaged. (A median form of averaging again has the merit of minimizing the effect of outliers). The shape factor values of the individual unit hydrographs are then scanned until the unit hydrograph is noted that has shape factors approximately equal to the average values. This is then adopted to represent the average catchment response, constituting in effect an average unit hydrograph.

Compared to ordinate by ordinate averaging, the shape factor demands rather more from the individual unit hydrographs. They must be stable and have comparable time bases. The former condition is desirable to ensure that the adopted unit hydrograph will itself be stable. But the over-riding need for both conditions to be met is that the shape factors of the unit hydrograph are sensitive to the extremities of the distribution and can take on unrealistic values if the unit hydrograph is oscillatory or has an abnormally short or long time base. The coefficients of skewness and peakedness are particularly susceptible in this respect.

Reed met these requirements by using the restricted least-squares method of unit hydrograph derivation (Section 2.4) and by constraining the time base of individual event unit hydrographs to a predetermined value. (The latter was achieved by linking the processes of rainfall and runoff separation and applying Equation 2.5 in reverse, ie to determine the duration of response runoff from the duration of net rainfall and the specified time base for the unit hydrograph). The present study utilised net rainfall and quick response runoff data prepared in Flood Studies format (see Section 1.2). Thus the shape factor averaging method was handicapped by the fixed time base requirement not being met.

3.4 Comparisons

3.4.1 Introduction

Average unit hydrographs were derived by the ordinate by ordinate averaging methods (Section 3.2) and the shape factor averaging method (Section 3.3) for each of the eight catchments under study. The ordinate by ordinate averaging methods were applied to unit hydrographs derived by each of the three procedures considered in Section 2. It was found that the relative properties of the ordinate by ordinate methods differed little with the type of unit hydrograph being averaged; thus only results for the least-squares procedure are quoted in subsection 3.4.2. Because the shape factor averaging method relies on the individual unit hydrographs being stable, comparisons made in subsection 3.4.3 relate to use of the restricted least-squares procedure.

3.4.2 Comparison of ordinate by ordinate methods

Figure 3.3 (opposite) shows average unit hydrographs derived by the mean and median methods for catchment 46003. The unannotated lines in this diagram are the individual unit hydrographs being averaged. The corresponding 'peaks aligned' results are not shown but Figure 3.4 allows direct comparison of all four average unit hydrographs derived.

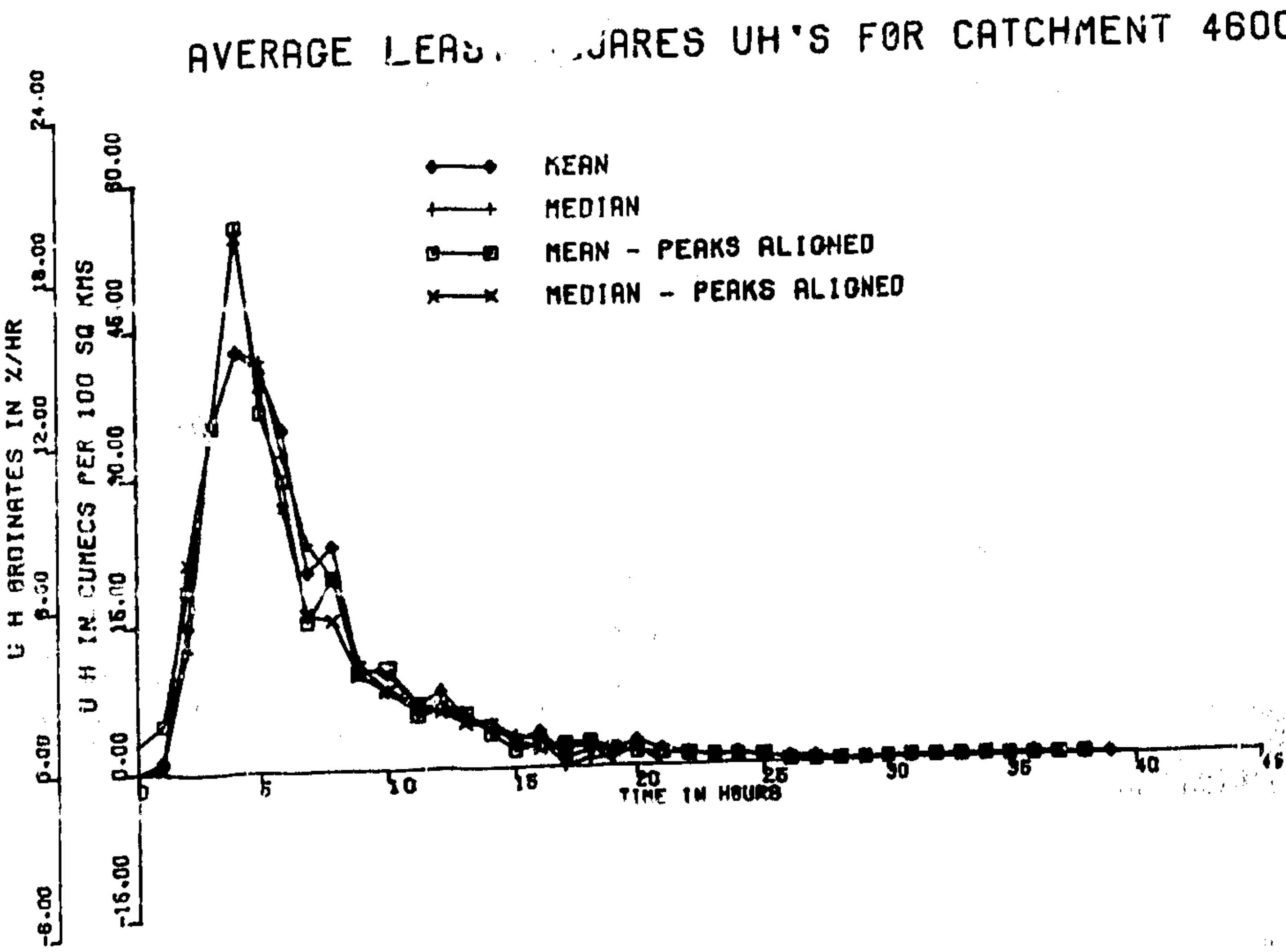


FIGURE 3.4 **Comparison of ordinate by ordinate averaging methods**

These results for catchment 46003 illustrate several properties of the methods. Firstly, peak alignment does of course lead to an average unit hydrograph with a higher and somewhat slimmer peak. A second feature to note is that the mean peaks aligned method produces an average unit hydrograph that rises before the time origin. This anomaly arises because one (or more) of the individual unit hydrographs inevitably has a longer than average time to peak.

A third point, not immediately obvious from Figure 3.4, is that on this catchment, both the median and median peaks aligned methods generate an average unit hydrograph that has significantly less than unit volume. For example, the median peaks aligned unit hydrograph for catchment 46003 has a volume of only 0.912. Not all unit hydrographs derived by the median peaks aligned method suffered such a volume loss, but even lower volumes were experienced on three of the seven other catchments.

Because this study has used real rather than synthetic data, there is no true average unit hydrograph against which to assess those derived by the various methods. In the authors' opinion, none of the ordinate by ordinate methods is totally satisfactory. The methods without peak alignment tend to produce a unit

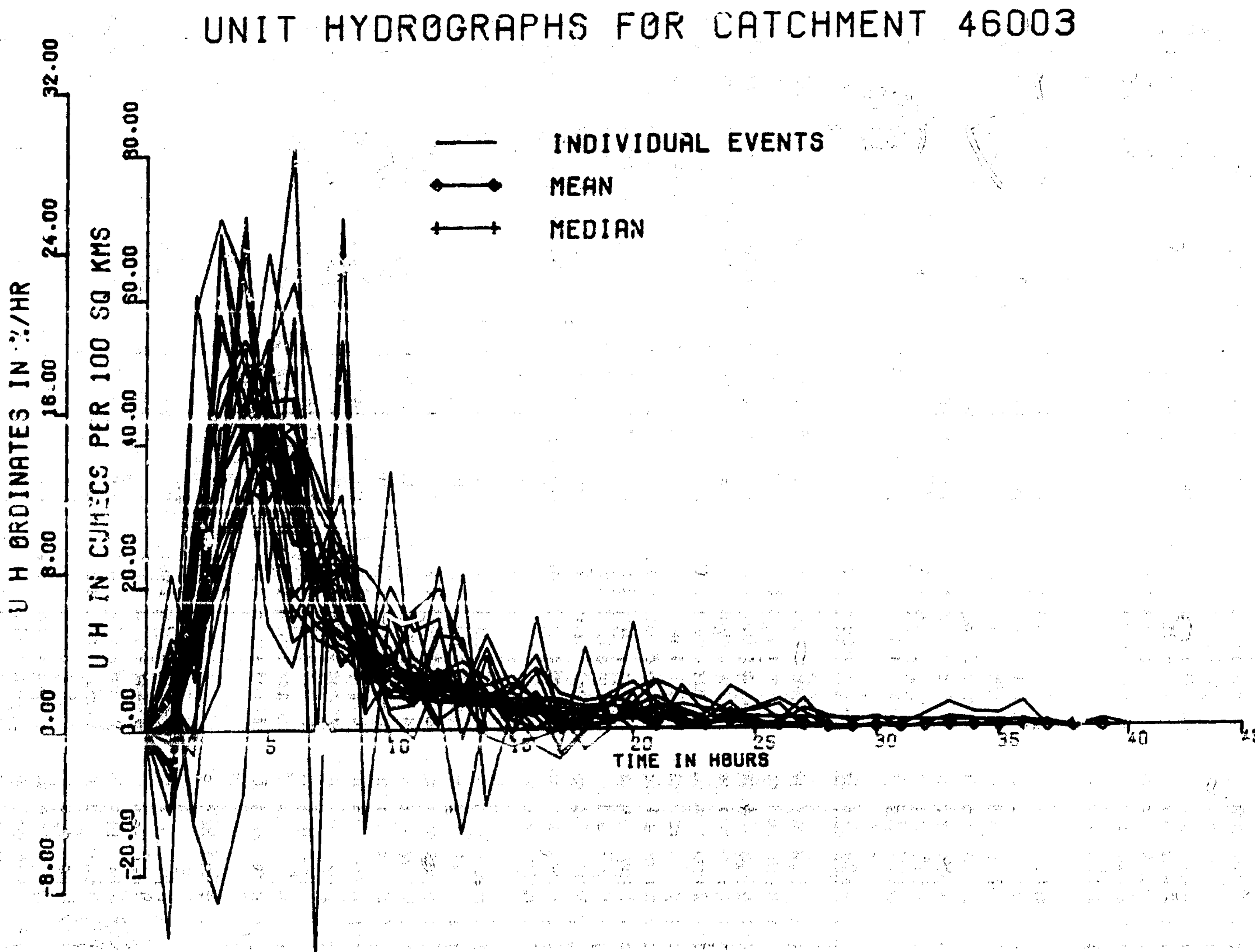


FIGURE 3.3 **Simple averaging of least-squares unit hydrographs** ***(see also Figure 3.4)***

hydrograph that is rather too diffuse when compared with the unit hydrographs being averaged. This effect is all the more noticeable when the individual unit hydrographs have a wide range of times to peak. The technique of peak alignment perhaps corrects this too far, producing an average unit hydrograph that if anything is rather too slim. Nor is there a clear preference between mean and median methods. Use of a median form has the advantage of minimizing the effect of outliers but the disadvantage of producing an average unit hydrograph that is sometimes deficient in volume. Comparisons with 'by eye' average unit hydrographs, derived using the textbook method (Section 3.1), suggested that the median peaks aligned technique was generally the most successful at preserving the characteristic shape of the unit hydrographs being averaged. For this reason alone, we favour use of the median peaks aligned technique over the other three ordinate by ordinate methods considered.

It was stated in Section 3.4.1 that the relative properties of the ordinate by ordinate methods differ little with the type of unit hydrographs being averaged. This is not to say that particular combinations of derivation and averaging techniques may not be particularly appropriate. For example, because the Flood Studies Report method of derivation has a tendency to underestimate peak ordinates of individual unit hydrographs, this effect might be offset by use of a peaks aligned averaging technique (which has a converse tendency).

3.4.3 Assessment of shape factor method

Average unit hydrographs determined by the shape factor method generally stood apart from those produced using ordinate by ordinate averaging. On most of the catchments considered, the choice of an individual unit hydrograph to represent the average response was less than easy, none of them having quite the right mix of shape factor values. Various strategies could be followed to overcome this difficulty. For example, a linear combination of a number of the individual unit hydrographs could be contrived to yield one with precisely the required shape factor values. Alternatively, at this stage in the analysis a synthetic unit hydrograph could be substituted, fitted in effect by the method of moments (Nash (1959)). However, such grand schemes might defeat what is otherwise an uncomplicated attitude to unit hydrograph averaging: namely to pick one from the bunch that is typical.

On catchment 65001 there was little difficulty in selecting a representative unit hydrograph; that derived from event 10 had shape factor values that closely approximated the corresponding median values for the 15 events (see Figure 3.5). Figure 3.6 contrasts this unit hydrograph with that determined by applying the median peaks aligned method of averaging to unit hydrographs derived by restricted least-squares. Perhaps the one noticeable difference is in the breadth of the upper half of the unit hydrographs. An alternative comparison is provided in Table 3.1 (page 20) in terms of shape factor values. These confirm that the major deviations between the two unit hydrographs are in terms of volume and spread, the latter being indexed by the coefficient of variation.

3.4.4 Conclusions

The median peaks aligned method is preferred amongst the four ordinate by ordinate averaging techniques considered, despite its tendency to produce a unit hydrograph with a volume appreciably less than unity. With regard to shape factor averaging it is difficult to draw firm conclusions because the rainfall and runoff separations used in this study were unsuited to the approach. Although it lacks total objectivity, shape factor averaging may be worth consideration, particularly in applications that seek to generalize unit hydrographs in terms of a small number of parameters.

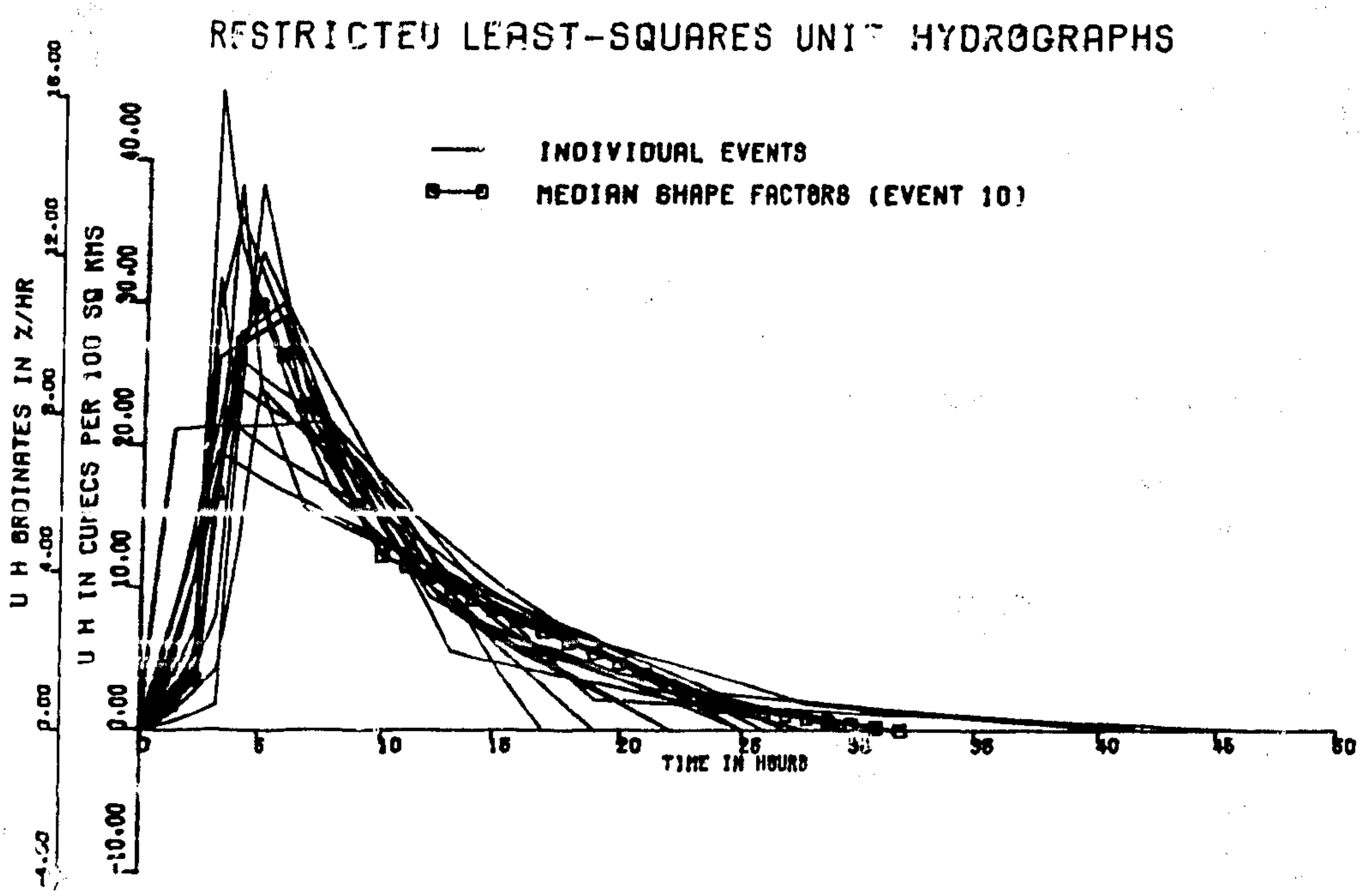


FIGURE 3.5 Shape factor averaging of restricted least-squares unit hydrographs

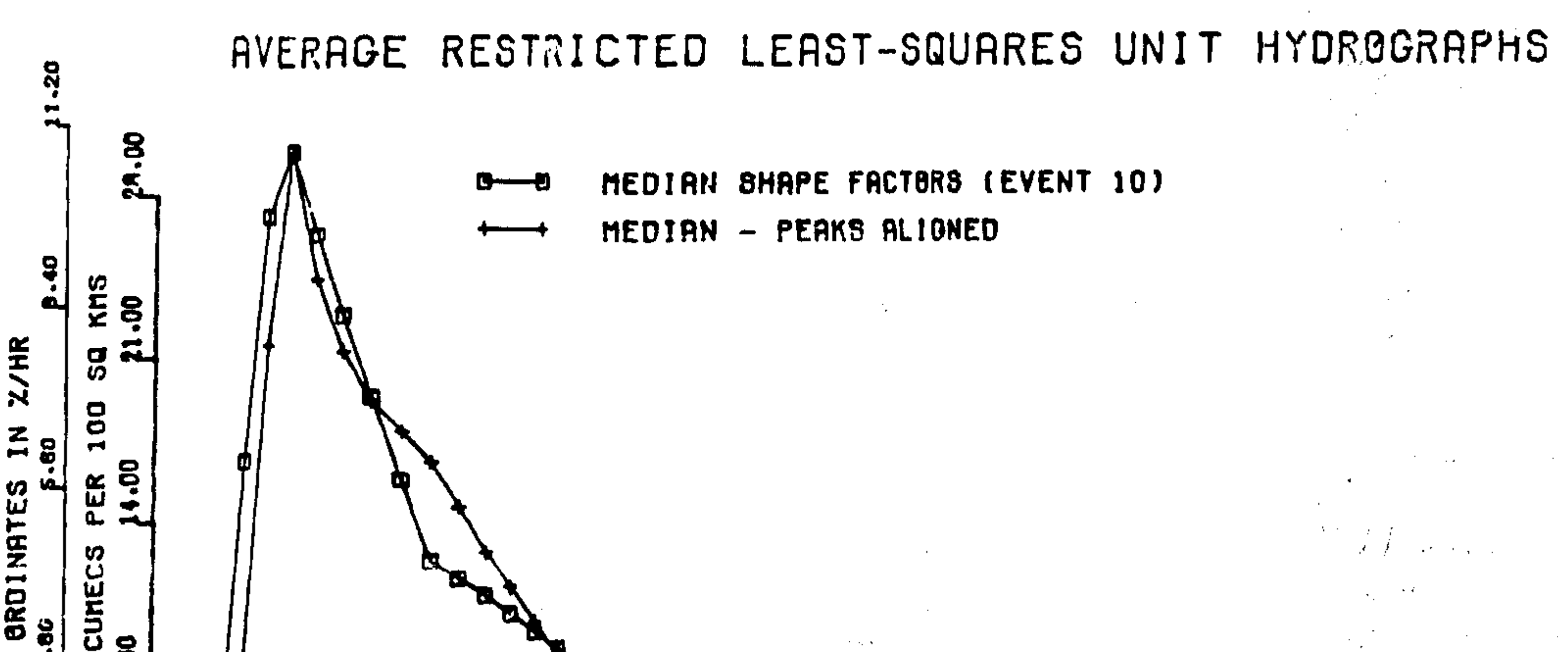


FIGURE 3.6 Comparison of shape factor averaging with an ordinate by ordinate method

TABLE 3.1 SHAPE FACTOR VALUES FOR RESTRICTED LEAST-SQUARES UNIT HYDROGRAPHS DERIVED ON CATCHMENT 65001

Symbol	Shape factor	Event 10	Median peaks aligned	Median values (15 events)
V	volume	0.987	0.929	0.987
M_1'	mean (hr)	9.72	9.76	9.44
c_2	coefficient of variation	0.606	0.531	0.597
c_3	coefficient of skewness	1.00	0.83	0.94
c_4	coefficient of peakedness	0.42	0.00	0.16

The one technique of averaging individual unit hydrographs carried forward, for comparison with joint analysis techniques, is the median peaks aligned method.

4 JOINT ANALYSIS TECHNIQUES

4.1 Introduction

The joint analysis of a number of events offers a possible short-cut to the derivation of an average unit hydrograph; it avoids the two-stage process of first deriving unit hydrographs (Section 2) and then averaging them (Section 3). Two methods of joint analysis are considered: event concatenation and event superposition.

4.2 Event concatenation

The possibility of chaining a number of events end to end and deriving a unit hydrograph from the resulting hypothetical event has been considered by Diskin and Boneh (1975) and others. This idea of event concatenation as a means of deriving an average unit hydrograph makes use of the assumption of time-invariance that is inherent in the adoption of a unit hydrograph based model. The principle of time-invariance states that the response to net rainfall is independent of the time at which the rainfall occurred; thus it is permissible to alter the time attached to an event provided that the relative timing of net rainfall and quick response runoff is preserved.

A simple example of event concatenation is shown in Figure 4.1.

4.3 Event superposition

The idea of superposing rather than concatenating events prior to analysis provided a particular stimulus to this study. The technique relies on the unit

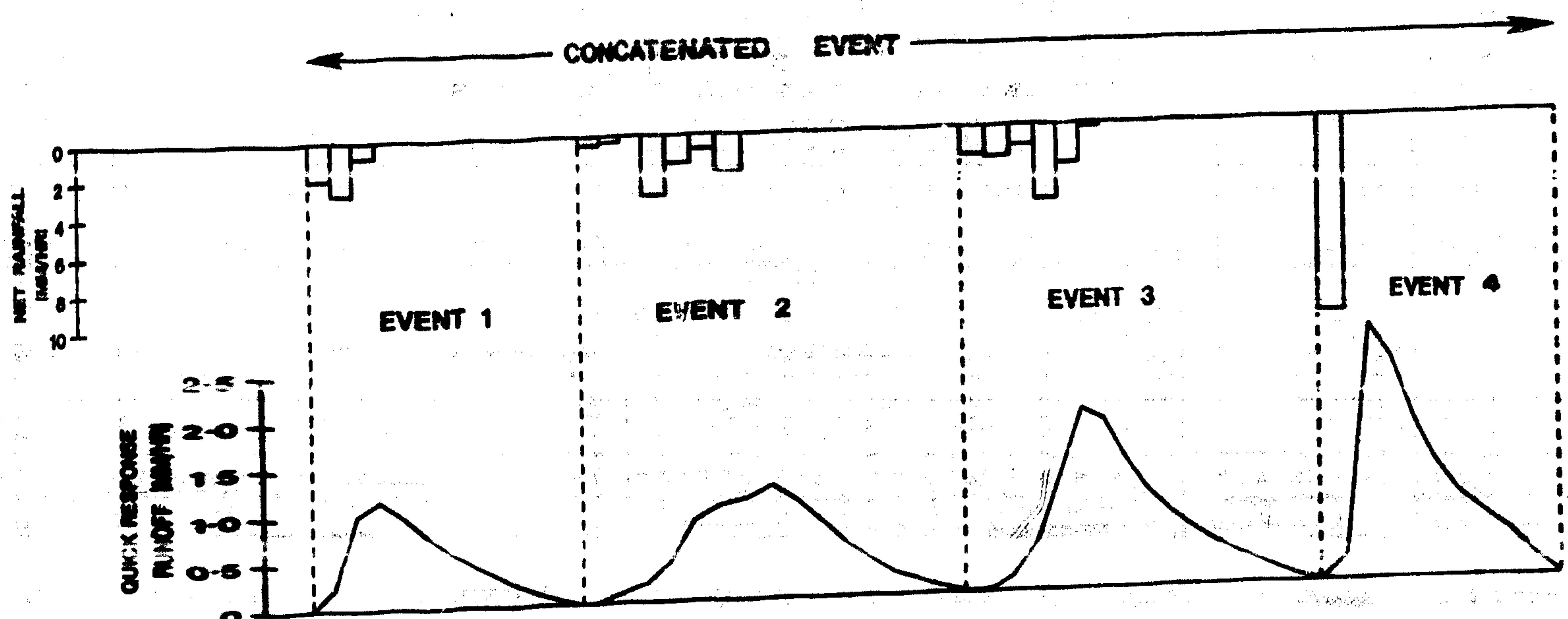


FIGURE 4.1 Event concatenation

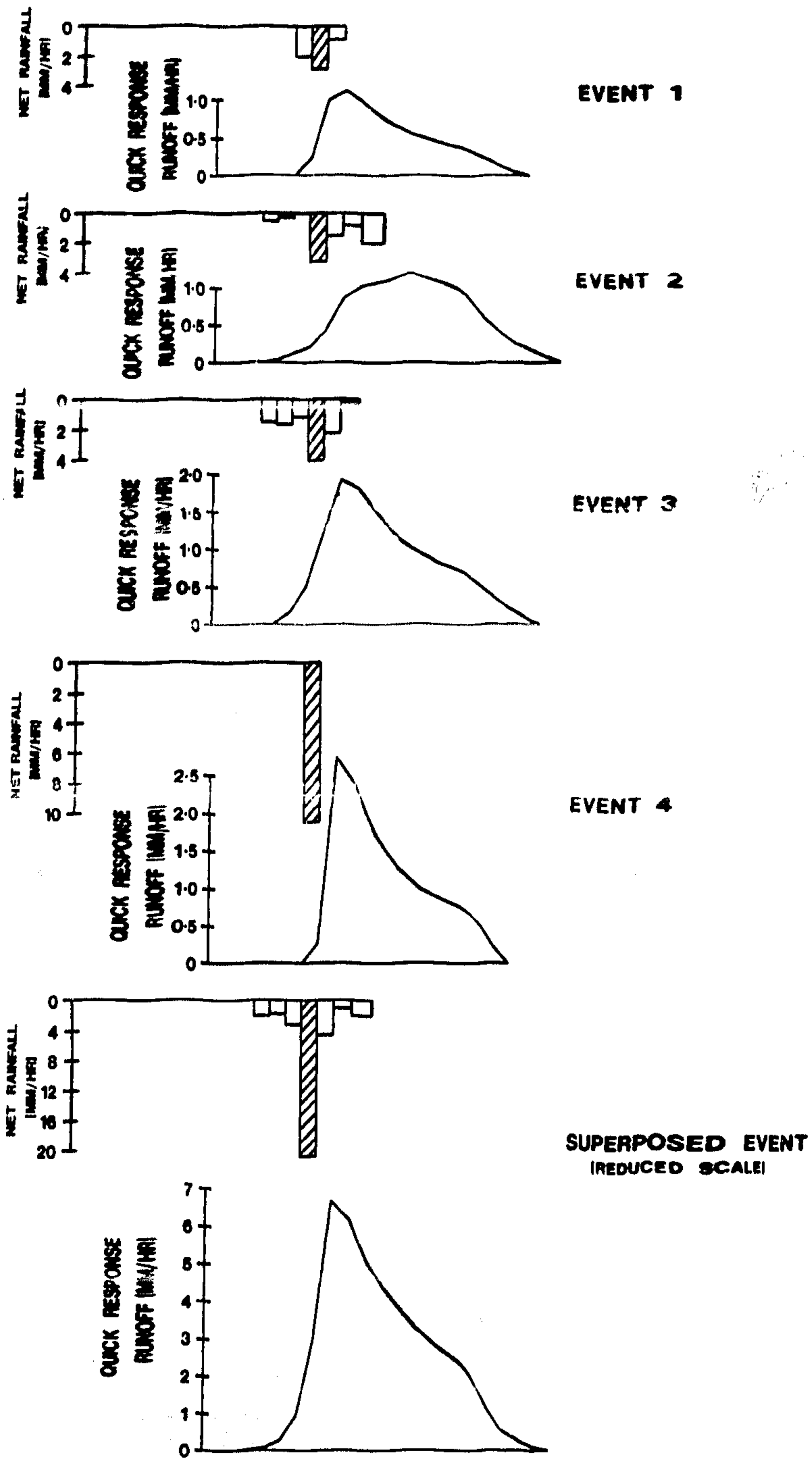


FIGURE 4.2 Event superposition

hydrograph assumptions of linearity and time-invariance. Originally the superposition was carried out by summating the event data in simple fashion: namely adding the first blocks of net rainfall together to form the first block of net rainfall in the superposed event, and so on. Subsequently it was realised that some systematic alignment of events prior to summation was advantageous. For reasons that will become clear in Section 6.3, the method finally adopted was to align the peak elements of net rainfall.

Figure 4.2 illustrates the superposition for the same example as that used to illustrate concatenation. Note that the alignments prior to summation preserve the relative timing of net rainfall and quick response runoff for each event.

4.4 Comparisons

Having combined individual events either by concatenation or superposition it is in principle possible to analyse the product by any method of unit hydrograph derivation. The comparisons here are made in terms of use of the least-squares method (Section 2.2).

Both techniques produced reasonably smooth unit hydrographs for each of the eight catchments studied and, in general, yielded very similar results. The main differences noted were in the tails of some of the unit hydrographs but it became clear that these arose primarily out of the different time bases used by the methods. The time bases were calculated according to Equation 2.5 but were, in effect, quite arbitrary; in the case of concatenation, the time base was that implied by the last event chained. To avoid clumsy comparisons the time bases of the joint analysis unit hydrographs were constrained to a particular value for each catchment, the value being determined as the upper quartile point of the distribution of time bases implied by the events on that catchment. (These constraints were applied during the unit hydrograph derivation process and were not accompanied by changes in the methods of rainfall and runoff separation used).

Figure 4.3 illustrates how the joint analysis methods performed on catchment 46003.

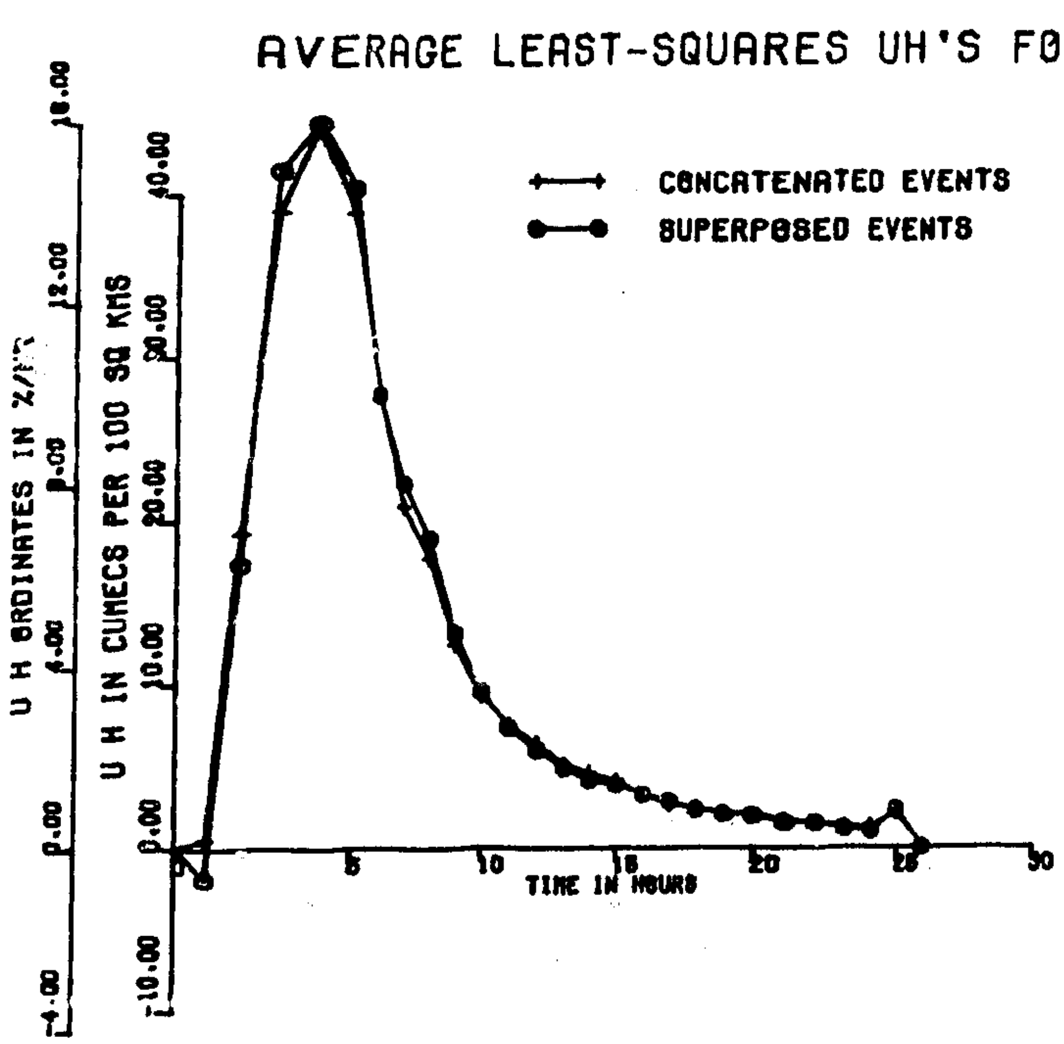


FIGURE 4.3

Comparison of joint analysis methods (catchment 46003)

There is little discernible difference between the unit hydrograph derived after event concatenation and that derived following superposition. The solutions differed most on catchment 58001 but even there the variation is minor (Figure 4.4).

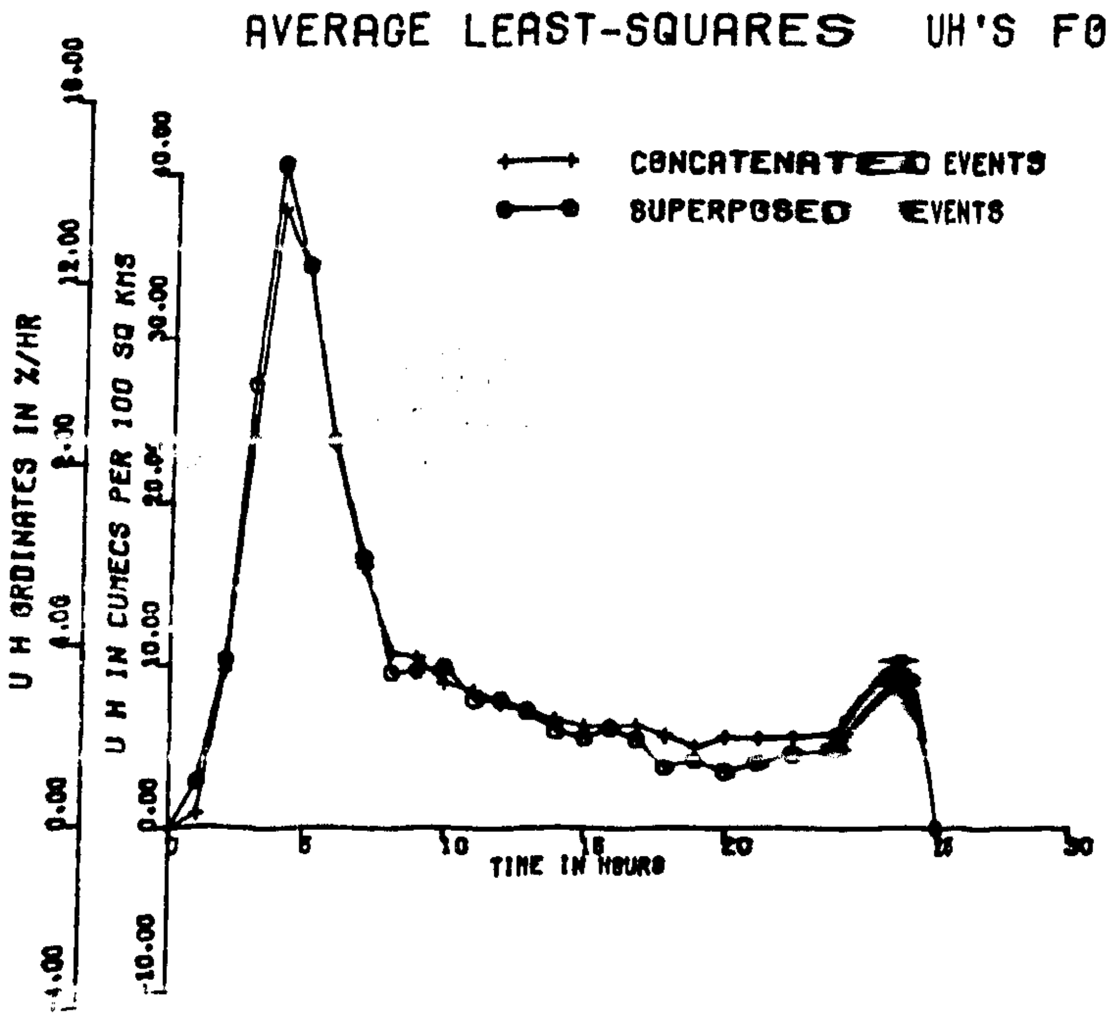


FIGURE 4.4

Comparison of joint analysis methods (catchment 58001)

However, in terms of computational effort the two methods differ greatly. Analysis of a concatenated event requires computation roughly proportional to the number of events chained and may, depending on the algorithm used to derive the least-squares solution, demand large computer storage. In contrast, a superposed event is scarcely longer than an individual event and poses no particular problem in analysis. In fact, as will be shown in Section 6, in many circumstances it is possible to use a simpler solution technique for a superposed event than can generally be used on an individual event. For these reasons, in the detailed comparisons that follow, the joint analysis approach to determining an average unit hydrograph is represented by the event superposition technique.

5 DETAILED COMPARISONS

5.1 Structure of comparisons

The three derivation methods considered (Section 2), the five averaging techniques (Section 3), and the two techniques of joint analysis (Section 4) combine to offer a multitude of ways of deriving an average unit hydrograph. Whilst it would be possible to make exhaustive comparisons of these combinations, it is perhaps more useful to draw on the comparisons already made and to consider further only a sub-set of methods. Although preferred methods emerged from Sections 3 and 4, the discussion in Section 2 must first be extended to consider if refinements of the basic least-squares method are of value when the goal is derivation of an average unit hydrograph (Section 5.2). Detailed comparisons are taken up in Section 5.3.

5.2 Value of refinements to the least-squares method

5.2.1 Introduction

In the analysis of individual events the unit hydrograph derived by the basic least-squares method is sometimes in need of smoothing. Of the two techniques of smoothing considered in Section 2, the restricted least-squares method appeared the more promising, the Flood Studies Report method having a tendency to attenuate the least-squares unit hydrograph. However, it is not clear whether refinements to the least-squares method are of value when the objective is derivation of an average unit hydrograph. The utility of the restricted least-squares and Flood Studies Report methods is evaluated in turn for the median peaks aligned and event superposition approaches to this problem.

5.2.2 When using the median peaks aligned technique

Figure 5.1 presents median peaks aligned unit hydrographs for catchment 46003 obtained with the least-squares, Flood Studies Report, and restricted least-squares methods. The first feature to note is that all three unit hydrographs are [illegible]ically stable. That smoothing of the unit hydrographs being averaged is perhaps superfluous is supported by the close agreement of the least-squares and restricted least-squares solution. The systematic bias imparted by the Flood Studies Report method is confirmed, the unit hydrograph being lower-peaked and more diffuse.

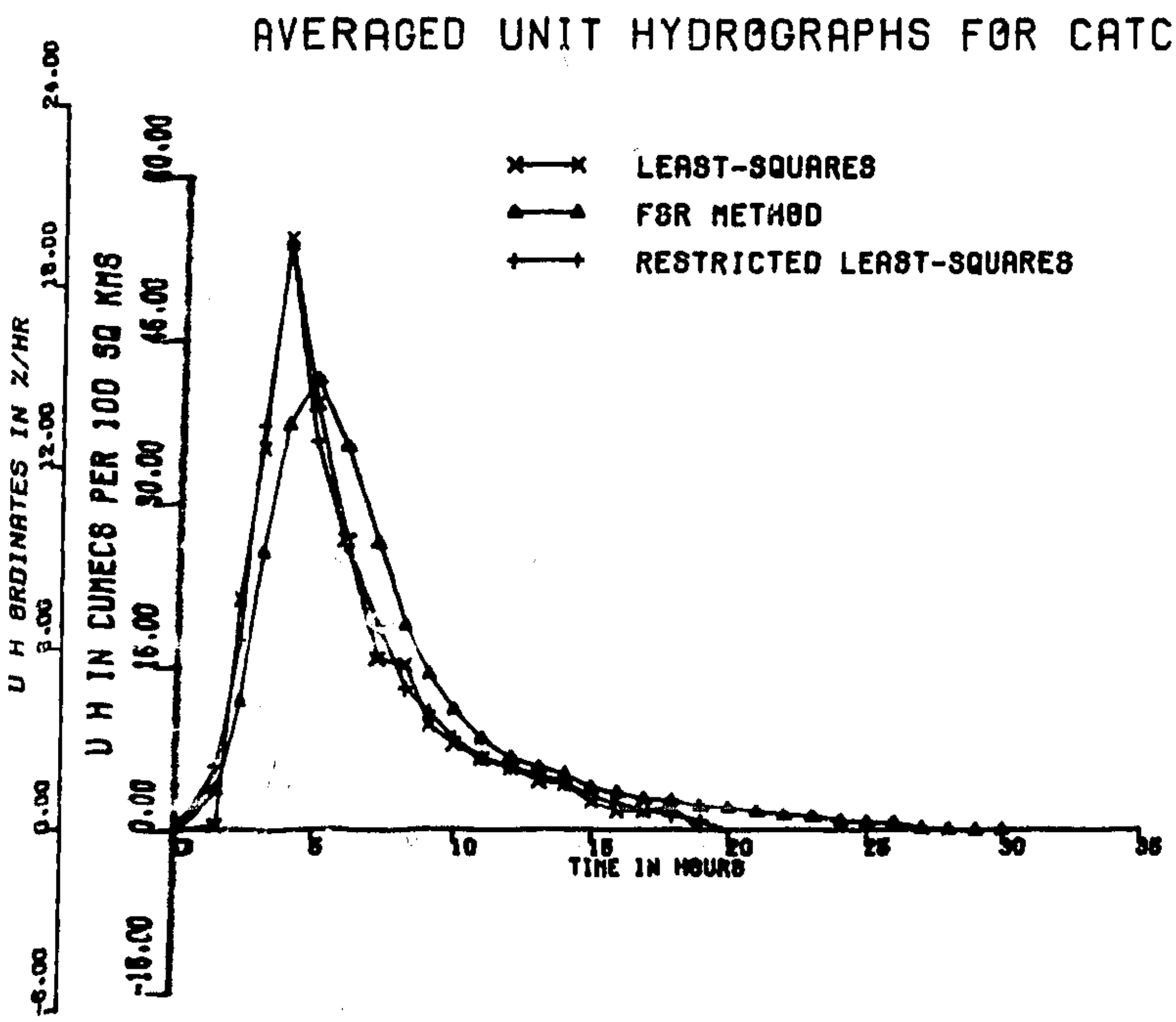


FIGURE 5.1 Comparison of derivation methods for median peaks aligned technique (catchment 46003)

Not all of the average unit hydrographs derived were quite as smooth as those illustrated in Figure 5.1 although they are by no means exceptional. On catchment 65001 (Figure 5.2) the average unit hydrograph derived by least squares is a little perturbed but scarcely to such an extent as to make smoothing essential.

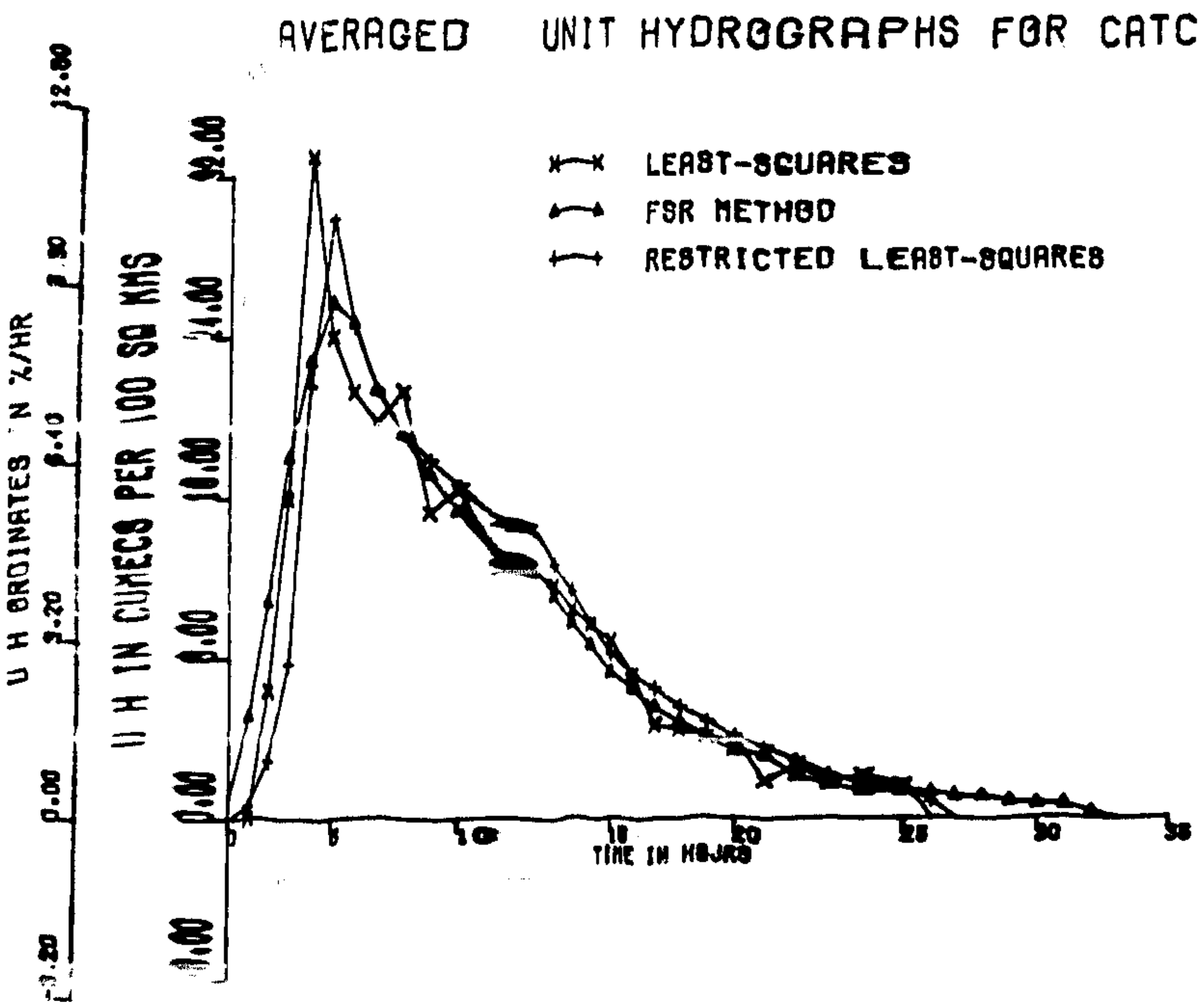


FIGURE 5.2

Comparison of derivation methods for median peaks aligned technique (catchment 65001)

5.2.3 When using the event superposition technique

Figure 5.3 shows average unit hydrographs obtained for catchment 46003 using the event superposition technique. Again, all three methods of derivation produce a

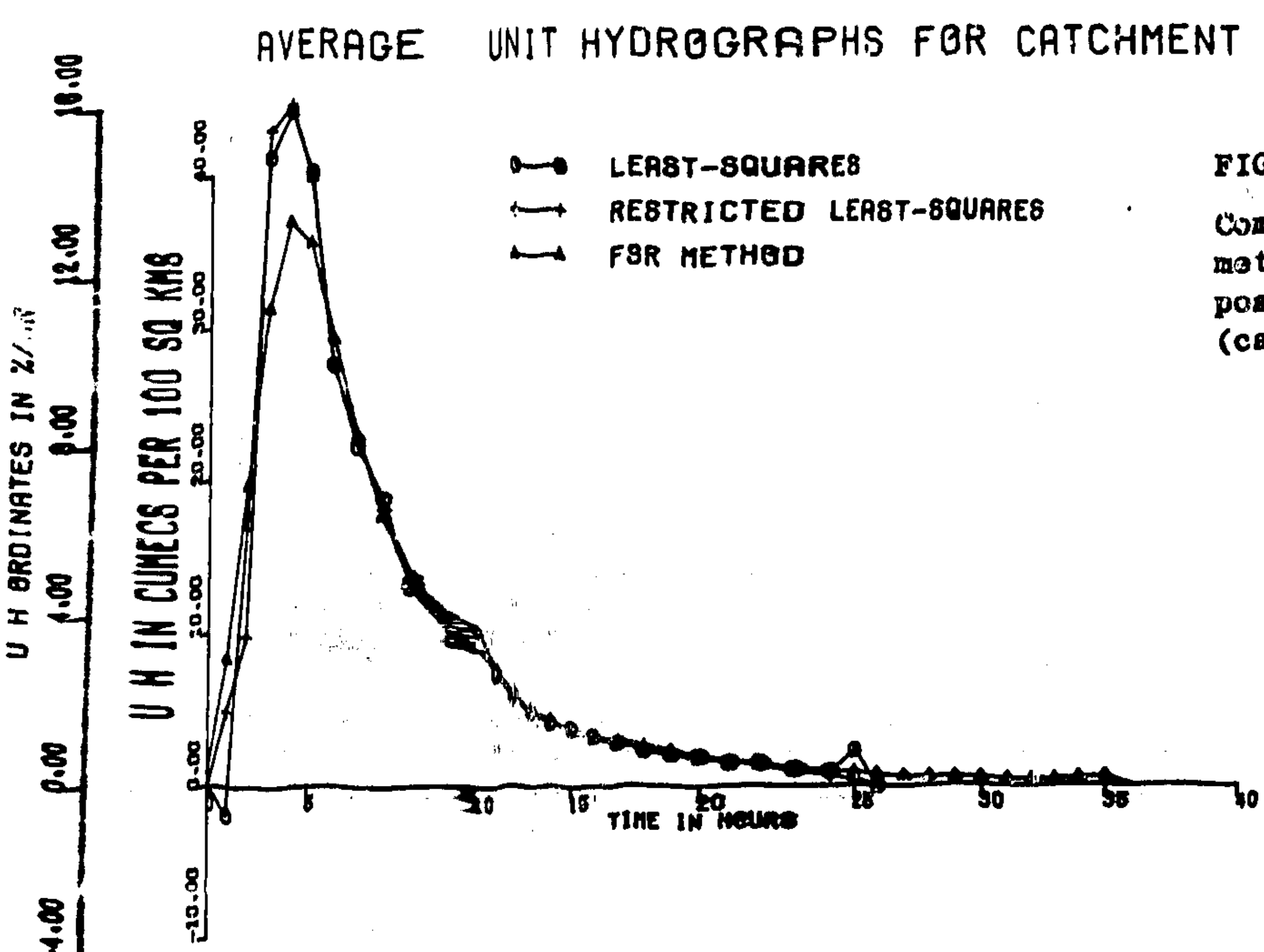


FIGURE 5.3

Comparison of derivation methods for event superposition technique (catchment 46003)

basically stable unit hydrograph for this catchment and the value of using restricted least-squares rather than least-squares is minimal. The bias introduced by the Flood Studies Report method is smaller than in the median peaks aligned case but still evident. For catchment 65001 (Figure 5.4) there is little to fault in the average unit hydrograph derived by least-squares.

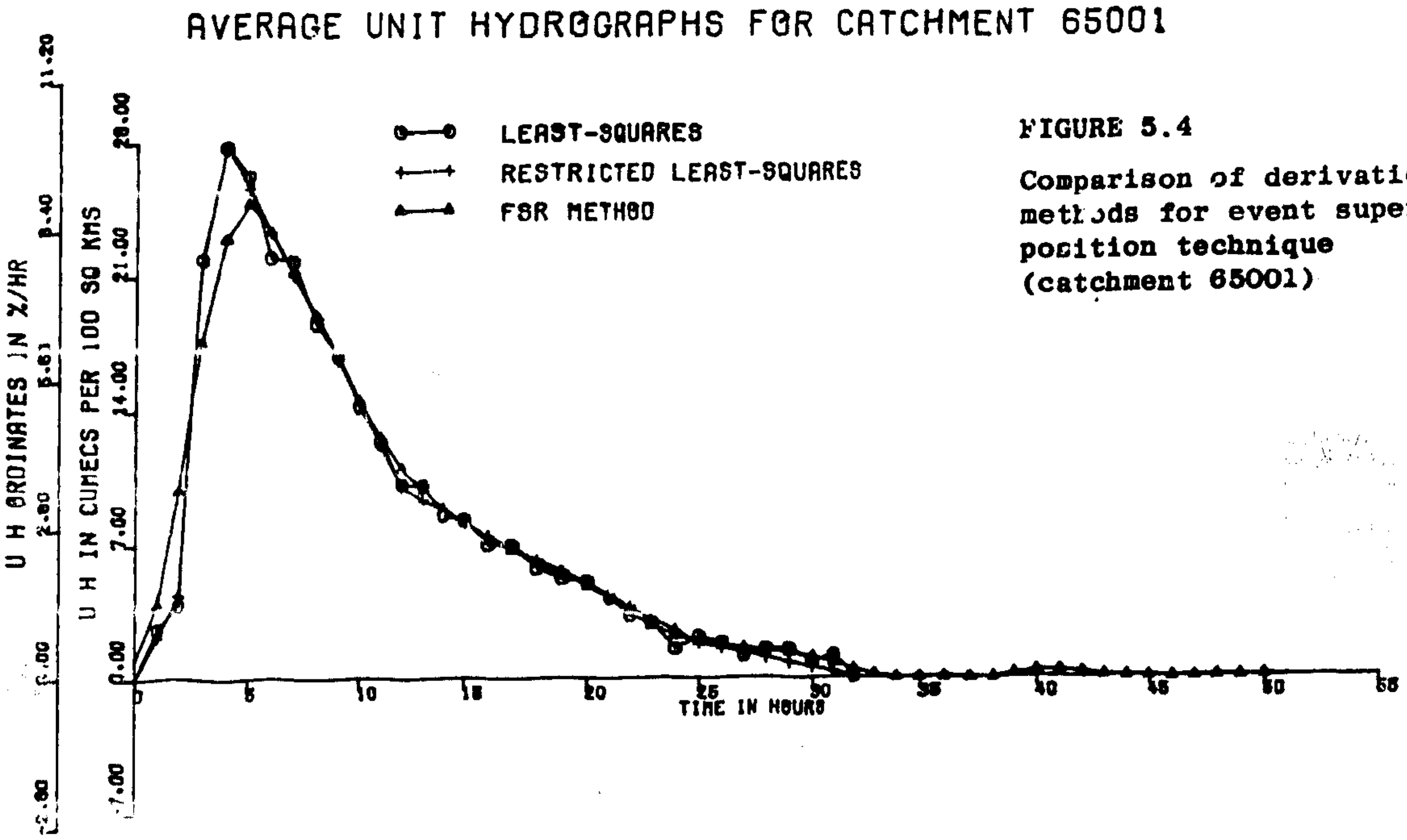


FIGURE 5.4

Comparison of derivation methods for event superposition technique (catchment 65001)

5.2.4 Conclusions

Drawing on the results obtained for all eight catchments it is concluded that the least-squares method generally produces an acceptably smooth average unit hydrograph, whether this be determined by the median peaks aligned technique or that of event superposition. By this criterion, refinements to the least-squares method seem to be of little value if a reasonable number of events are available for analysis. The Flood Studies Report method yields average unit hydrographs that are consistently lower-peaked and more diffuse than those obtained by the other two methods.

5.3 Median peaks aligned or event superposition?

5.3.1 Introduction

So far in this report comparisons have been made and methods set aside without recourse to formal testing of derived unit hydrographs. The median peaks aligned and event superposition techniques present alternative ways of deriving an average unit hydrograph and, as will be shown in the next subsection, yield generally distinct results. The question arises, does one technique consistently produce a better or more useful representation of the average response than the other?

5.3.2 Visual comparison of derived unit hydrographs

In what follows MEDPA denotes the median peaks aligned technique and SUPER denotes event superposition. Figure 5.5 shows average unit hydrographs derived for catchment 46003 by MEDPA and SUPER; as throughout this section, comparisons are based on results obtained using the least-squares method. It is seen that there is an appreciable difference between the unit hydrographs. The variation on catchment 65001 (Figure 5.6) is rather less but similar in that MEDPA again produces a higher-peaked and somewhat slimmer unit hydrograph; and this difference was evident on all but one of the eight catchments.

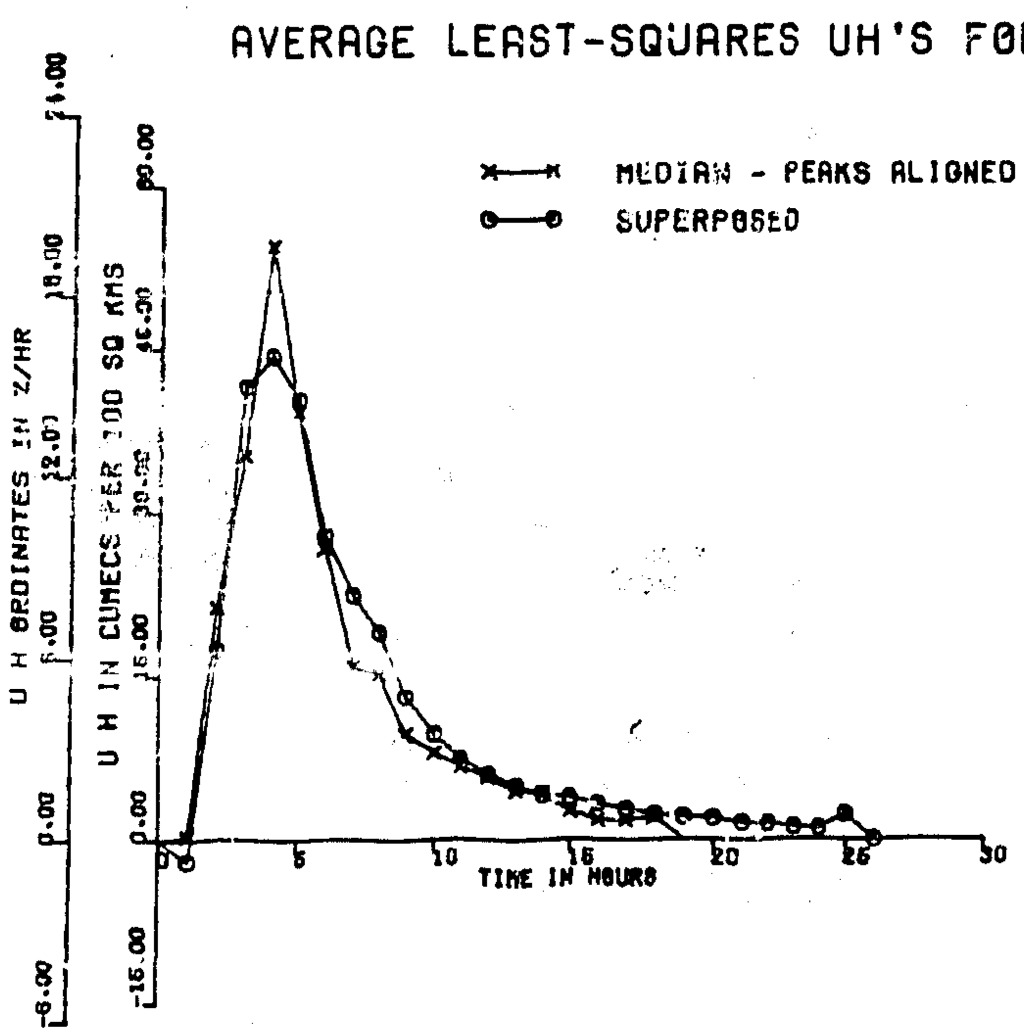


FIGURE 5.5

Median peaks aligned and event superposition average unit hydrographs for catchment 46003

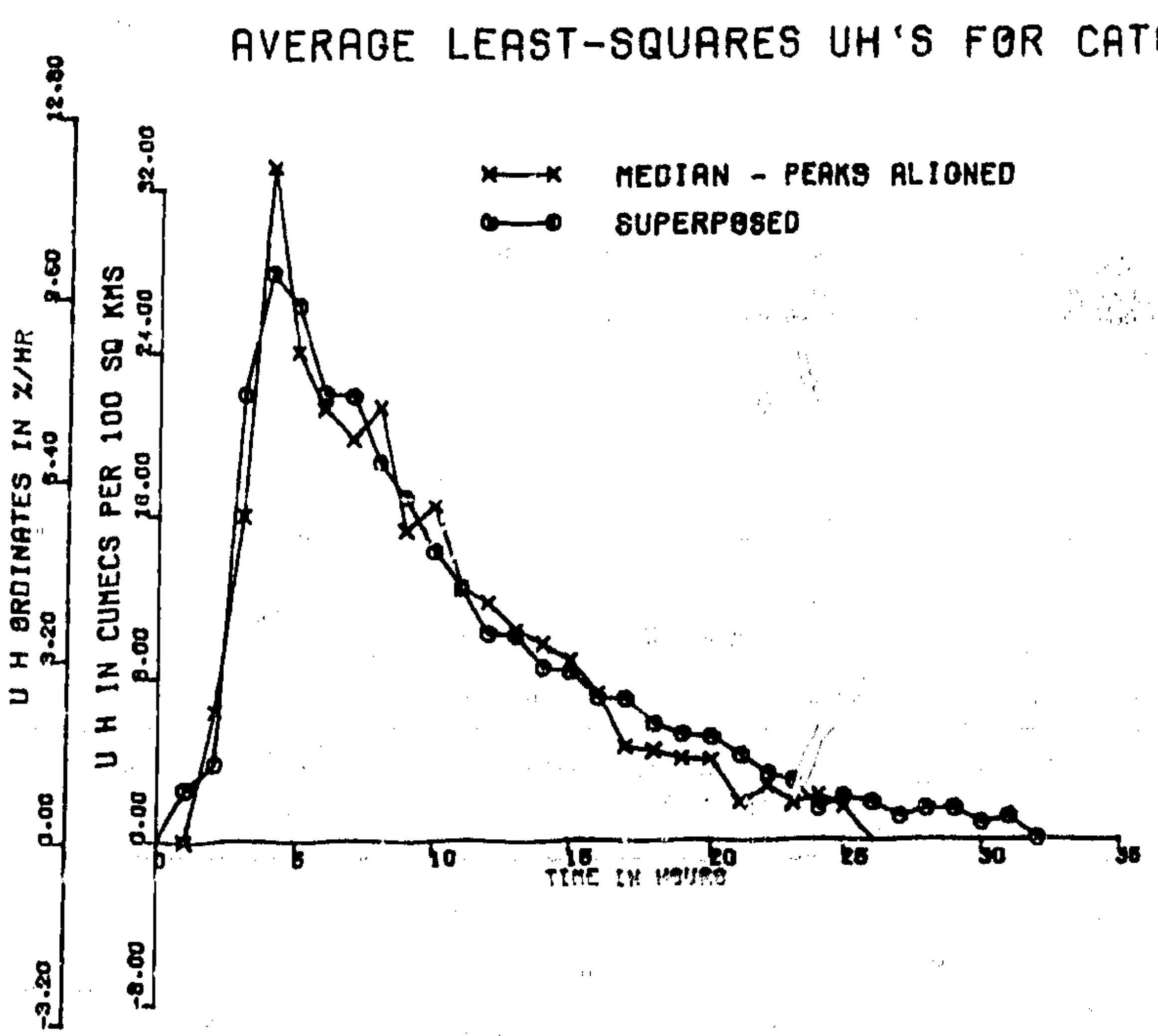


FIGURE 5.6

Median peaks aligned and event superposition average unit hydrographs for catchment 65001

A drawback of MEDPA noted earlier is that the median form of averaging often results in a unit hydrograph that has a volume significantly less than unity (Subsection 3.4.2). Before applying such a unit hydrograph it is clearly desirable to make an adjustment, the obvious one being to scale to unit volume. The relevance is that this adjustment will in general accentuate the difference between MEDPA and SUPER results. Reference to a scaled version of the median peaks aligned unit hydrograph will be made by the extended acronym MEDPAS.

5.3.3 Shape factor comparison of derived unit hydrographs

Using shape factors (see Section 3.3 and Appendix 2) an attempt was made to assess which of MEDPA and SUPER generally produced an average unit hydrograph more typical of the individual unit hydrographs. Characterizations of the MEDPA and SUPER unit hydrographs are given in columns 3 and 4 of Table 5.1 for catchment 46003. These statistics confirm that the major differences are in terms of volume and spread. The former is dealt with by use of MEDPAS rather than MEDPA but the latter (indexed by the coefficient of variation) remains as a statement that the MEDPA unit hydrograph is peakier than the SUPER one.

Following the first step of the shape factor averaging approach outlined in Section 3.3, median values of the shape factors of the individual unit hydrographs were calculated and are given in column 5 of Table 5.1 for comparison with the MEDPA and SUPER shape factor values. (As explained in Section 3.3 it is preferable to use the restricted least-squares method when a shape factor representation is sought; however, the required comparison is of average unit hydrographs derived by least-squares). It is seen from Table 5.1 that the median values of the individual event shape factors (column 5) fall part way between the respective MEDPA and SUPER values. If it is assumed that the median shape factor representation provides a neutral method of averaging then the conclusion to be drawn is that MEDPA tends to underestimate the mean, spread and skewness inherent in the individual unit hydrographs whereas SUPER tends to overestimate these features. Results obtained for six of the other seven catchments analysed followed a general pattern similar to that detailed in Table 5.1.

TABLE 5.1 SHAPE FACTOR VALUES FOR UNIT HYDROGRAPHS DERIVED FOR CATCHMENT 46003

(1)	(2)	(3)	(4)	(5)
Symbol	Shape factor	Median peaks aligned of least-squares u.h's (MEDPA)	Event superposition and least-squares (SUPER)	Median values (22 events)
V	Volume	0.912*	1.006	0.986
M_1'	Mean	5.85	6.97	6.72
c_2	Coefficient of variation	0.559	0.669	0.577
c_3	Coefficient of skewness	1.27	1.71	1.47
c_4	Coefficient of peakedness	1.22	2.90	1.56

**1.000 after scaling*

5.3.4 Comparison by reconstruction of response hydrographs

In terms of answering the question posed in subsection 5.3.1 the comparisons presented in subsections 5.3.2 and 5.3.3 are inconclusive; there is no clear reason to prefer either the median peaks aligned technique or the event superposition technique. It was decided that if a firm evaluation were to be reached then it would have to be based on some form of objective testing of derived unit hydrographs.

The procedure used to test the MEDPAS and SUPER unit hydrographs was as follows. For each catchment, and for both types of unit hydrograph, the response hydrographs were reconstructed from the net rainfall data and compared with the response runoff data. Before discussing these comparisons it is perhaps as well to say a few words of justification regarding use of the same data in testing as in derivation of the unit hydrographs. The point to appreciate is that it is the performance of methods of determining an average unit hydrograph that is under scrutiny, not the performance of a rainfall/runoff model based on such a unit hydrograph.

Notwithstanding the above, because the same data are used in testing as in fitting it is as well to assess performance in terms of a feature not explicitly minimized in the fitting process (ie other than least-squares). The feature chosen was the percentage discrepancy between the reconstructed peak response runoff and the 'observed' peak response runoff. Mean values of the percentage error so defined are listed in Table 5.2 for each catchment and for reconstructions using MEDPAS and SUPER unit hydrographs. It is clear from the tabulations that use of MEDPAS tends to overestimation of peak response runoff and that use of SUPER tends to underestimation. The final row in Table 5.2 indicates that the mean percentage error, on all 125 events considered, is approximately +6% for MEDPAS and -6% for SUPER. This symmetry about a zero mean percentage error is emphasised by a further statistic; of the 125 events subject to reconstruction, peak response runoff estimation was more accurate by MEDPAS on 63 events but by SUPER on the remaining 62.

TABLE 5.2 TESTING [illegible] RECONSTRUCTION OF RESPONSE [illegible] (ALL EVENTS)

Catchment number	No. of events	Mean % error in peak response runoff	
		MEDPAS	SUPER
23002	10	2.28	- 8.98
5004	11	12.56	- 3.49
46003	22	4.07	- 9.05
53005	14	1.82	-14.05
58001	17	9.43	- 7.68
61001	21	15.02	- 3.50
65001	15	5.94	- 0.88
76014	15	-4.86	- 0.2[illegible]
All catchments	125	6.13	- 5.96

Having failed a third time to elect a leader, a final check was made on how the two techniques fared in reconstructing the major events. On each catchment the events were ranked using peak response runoff as a measure of event size. The aim was to run the test on the three biggest events on each catchment but the procedure was relaxed to cater for the fact that on one catchment there were only two outstanding events whereas on others there were four. In all, 26 events were thus selected. Examination of the mean values of the percentage error in peak response runoff (Table 5.3) at last yielded a marked preference for one technique, namely SUPER. The comparison revealed that SUPER led to a superior estimate of peak response runoff for 19 of the 26 events.

TABLE 5.3 TESTING BY RECONSTRUCTION OF RESPONSE RUNOFF (MAJOR EVENTS)

Catchment number	No. of events	Mean % error in peak response runoff	
		MEDPAS	SUPER
23002	3	- 8.01	-14.33
45004	3	10.16	- 8.64
46003	4	6.28	- 7.19
53005	2	9.27	- 2.35
58001	4	22.11	3.05
61001	4	22.71	6.80
65001	3	12.86	6.79
76014	3	-20.00	-16.53
All catchments	26	8.00	- 3.55

Perhaps this boost for event superposition is by chance but there is a logical explanation why SUPER can be expected to perform rather better on the bigger events. The reason is that the quantity minimized in deriving a unit hydrograph by event superposition is the error sum of squares of the superposed flows - to which big events contribute proportionately more than little ones. This has the effect of biasing the derived unit hydrograph towards good reconstruction of the bigger events, as indeed is the case in the event concatenation technique (Section 4.2). In contrast, the median peaks aligned technique gives no special weight to unit hydrographs derived from big events when constructing its average unit hydrograph.

5.3.5 Summary

The tendency for MEDPAS to lead to overestimates, and SUPER to underestimates of peak response runoffs confirms the earlier finding that on average the MEDPAS unit hydrograph is too peaky and the SUPER one is too flat. Several strategies might be based on these results. A conservative assumption might be to adopt the MEDPAS technique because its use will tend to overestimation rather than underestimation of flows. An opposite conclusion but one based on a similar premise would be to favour SUPER because it is weighted towards the response observed in the bigger events. A third strategy might be to adopt the average of the MEDPAS and SUPER unit hydrographs! One final conclusion that might be drawn from the comparisons is that there is need of a better method of unit hydrograph averaging, perhaps based on the shape factor averaging approach.

But perhaps it is time to face the question that puts this study into perspective - does the choice of method matter?

5.4 Does it matter?

A common misunderstanding is to think that a 20% increase in the peak of the unit hydrograph will generally lead to an increase of similar magnitude in the peak response runoff. This is only true in the very rare, and usually non-critical, case of an event with a net rainfall duration equal to the period of the unit hydrograph (eg 1 hour for the unit hydrographs considered in this report). More usually the duration of net rainfall spans several such periods and an averaging-out takes place. To illustrate the point, Figure 5.7 shows the response hydrographs reconstructed for event 9 on catchment 46003 using the MEDPAS and SUPER techniques. Compared with the difference in unit hydrographs evident in Figure 5.5 it is seen that the reconstructed response hydrographs differ much less. For the eight catchments considered, the peak ordinate of the MEDPAS unit hydrograph was, on average, 25.2% higher than that of the SUPER unit hydrograph. In contrast, the catchment average value of peak reconstructed response runoff was, on average, only 12.7% higher using the MEDPAS unit hydrograph than for the SUPER unit hydrograph. A lesson of this averaging-out phenomenon is that the peak ordinate of a unit hydrograph is not in itself a particularly effective measure on which to base studies of unit hydrograph variation. What is rather more important is the shape and magnitude of the unit hydrograph in the vicinity of the peak. Thus, the precise choice of method for determining an average unit hydrograph is unlikely to be crucial to the success of a rainfall/runoff modelling application. Indeed it should be remembered that the unit hydrograph forms only a part of a rainfall/run-off model in which, from the authors' experience, the major errors stem from estimation of net rainfall.

However, particular derivation techniques do have particular advantages. Methods which begin by deriving unit hydrographs from individual events provide an opportunity to check for systematic variation of unit hydrographs (ie between events) and allow a more thorough validation of the basic data, for example with respect to timing problems. On the other hand, as we shall see in the next section, the short-cut offered by the event superposition technique has attractions.

6 A SIMPLE METHOD

6.1 Introduction

As has been stated by Bree (1978) and confirmed in the present study, the general effect of joint analysis of a number of events is to reduce problems of instability in unit hydrograph derivation. For this reason, when derivation of a catchment average unit hydrograph is the goal, there is little need for the more complicated methods designed to provide smoothing. On the grounds of computational economy and relative simplicity we suggest that event superposition followed by the basic least-squares method is a particularly effective way of arriving at an average unit hydrograph.

Perhaps with modern computer power these reasons for favouring superposition and

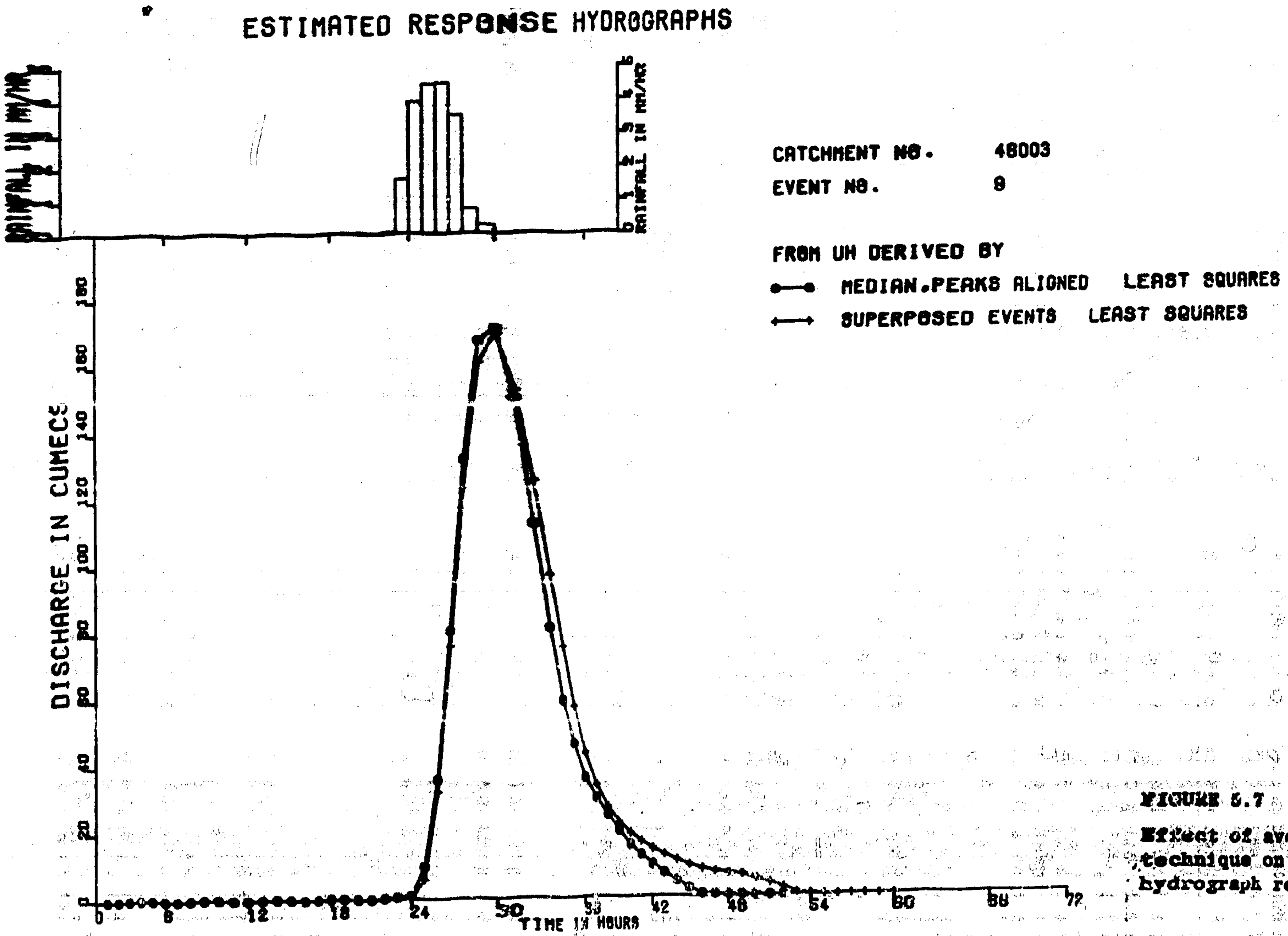


FIGURE 5.7

Effect of averaging technique on response hydrograph reconstruction

least-squares are slim. Those specialists who already have an effective means of unit hydrograph derivation to hand may well find this report of interest rather than of consequence. But in the expectation that it will be read by others less committed we believe that it is worth discussing a simple variant of the preferred method that brings derivation of an average unit hydrograph within the scope of pocket calculator implementation. The key to the simplification is abandonment of the least-squares criterion in favour of an iterative solution technique.

6.2 Iterative methods of unit hydrograph derivation

Iterative methods have long been used for unit hydrograph derivation, a well-known version being Collins' method (see, for example, Wilson (1974)). What is less well-known is that in certain circumstances the iterative methods may diverge.

Consider the case of a rainfall event that is dominated by a particular block of net rainfall, x_p. Then the coefficient matrix in the discrete convolution relation, (Equation 3.4), is dominated by the terms x_p which appear on the p'th diagonal:

$$\left|\begin{array}{ccccc} x_1 & & & & \\ \cdot & x_1 & & & \\ \cdot & \cdot & \cdot & & \\ x_p & \cdot & \cdot & \cdot & \\ \cdot & x_p & \cdot & \cdot & x_1 \\ x_N & \cdot & \cdot & \cdot & \cdot \\ & x_N & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & x_p \\ & & & \cdot & \cdot \\ & & & & x_N \end{array}\right| \left|\begin{array}{c} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{array}\right| = \left|\begin{array}{c} y_1 \\ \cdot \\ \cdot \\ y_p \\ \cdot \\ \cdot \\ \cdot \\ y_{p+n-1} \\ \cdot \\ y_m \end{array}\right| \qquad 6.1$$

If the equations that do not involve x_p are discarded then Equation 6.1 is reduced to the square system:

$$\left|\begin{array}{ccccc} x_p & \cdot & \cdot & x_1 & \\ \cdot & x_p & \cdot & \cdot & x_1 \\ x_N & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot \\ & x_N & \cdot & x_p & \end{array}\right| \left|\begin{array}{c} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{array}\right| = \left|\begin{array}{c} y_p \\ \cdot \\ \cdot \\ \cdot \\ y_{p+n-1} \end{array}\right| \qquad 6.2$$

or

$$X^*\underline{u} = \underline{y}^* \qquad 6.3$$

where X* is n x n and $\underline{y}^*$ is n x 1. The discarded equations correspond to runoff

observations before and after the main runoff effect (generated by x_p) and hence their neglect is of little concern. However, the importance of the case under consideration stems not from the fact that the 'discard excess equations' approach is a satisfactory alternative to using a least-squares criterion. Rather it is because the reduced system of equations (Equation 6.2) is amenable to an iterative solution process. Processes such as Jacobi or Gauss-Seidel iteration are simple in operation and, in cases where convergence is rapid, are very efficient solution methods.

The iterative processes are set up by dissecting the coefficient matrix X*, rearranging Equation 6.3 so that terms involving $\underline{u}$ appear on both sides of the equation, and by introducing an iteration count.

Let $$X^* = x_p I + L + U \qquad 6.4$$

represent the dissection of X* into a diagonal matrix $x_p I$, a lower triangular matrix L, and an upper triangular matrix U. Then Equation 6.3 can be written in the form:

$$\underline{u} = \frac{1}{x_p}\{\underline{y}^* - L\underline{u} - U\underline{u}\} \qquad 6.5$$

Introducing an iteration count, r, the Jacobi scheme for solving Equation 6.3 is denoted by:

$$\underline{u}^{(r+1)} = \frac{1}{x_p}\{\underline{y}^* - L\underline{u}^{(r)} - U\underline{u}^{(r)}\} \qquad 6.6$$

where $\underline{u}^{(r)}$ represents the r'th approximant to the solution $\underline{u}$. The Gauss-Seidel method differs in that new approximations are used as soon as they become available within each iteration, the scheme being written:

$$\underline{u}^{(r+1)} = \frac{1}{x_p}\{\underline{y}^* - L\underline{u}^{(r+1)} - U\underline{u}^{(r+1)}\} \qquad 6.7$$

Given an initial estimate, $\underline{u}^{(o)}$, Equation 6.6 or Equation 6.7 is used repeatedly to generate a sequence of approximants $\underline{u}^{(1)}$, $\underline{u}^{(2)}$ etc to the solution $\underline{u}$. The iterative process is said to converge if $\underline{u}^{(r)} \rightarrow \underline{u}$ as $r \rightarrow \infty$.

Theoretical necessary and sufficient conditions for convergence of the iterative processes exist but are not particularly helpful. The important practical criterion is that the coefficient matrix, X*, should be diagonally dominant and from Equation 6.2 it can be seen that this is assured by the dominance of the peak net rainfall block, x_p. It can be stated categorically that both the Jacobi and Gauss-Seidel processes will converge to solve Equation 6.3 if:

$$x_p > 50\% \text{ of } \sum_{i=1}^{N} x_i \qquad 6.8$$

From trials involving the derivation of unit hydrographs from each of the 125 events in the study dataset, it was confirmed that the Gauss-Seidel process is generally more effective than the Jacobi process. The latter rarely converged unless Condition 6.8 was satisfied; moreover, convergence of the Jacobi process was often slow if the peak net rainfall block was only mildly dominant (say, 50% of $\sum_{i=1}^{N} x_i < x_p < 75\%$ of $\sum_{i=1}^{N} x_i$). In contrast, convergence of the Gauss-Seidel process was generally much more rapid and occurred in many cases where Condition 6.8 was not satisfied. The convergence performance of the Gauss-Seidel method is summarised in Table 6.1 for the 125 unit hydrograph derivations from individual events.

TABLE 6.1 CONVERGENCE RESULTS FOR UNIT HYDROGRAPH DERIVATION BY GAUSS-SEIDEL ITERATION (125 INDIVIDUAL EVENTS)

Dominance of peak net rainfall block in range	Number of events for which Gauss-Seidel process:			
	converged* within 10 iterations	converged* in between 10 and 20 iterations	appeared set to converge	clearly diverged
0% - 50%	20	18	21	41
50% - 100%	25	1	2	0
Total	45	19	23	41

*Convergence deemed when none of unit hydrograph ordinates changed by more than 5×10^{-5} between iterations (ordinates sum to near unity).

6.3 Application to superposed events

The principle of event superposition was explained in Section 4.3. It is clear from the preceding discussion of iterative solution processes that a particularly useful way of superposing events is to align the peak net rainfall blocks prior to summation, so as to foster the desired property of a dominant net rainfall block in the superposed event. Whether at least 50% dominance is attained will depend on several factors, notably the number of individual events available for superposition. One hindrance may be the choice of an unnecessarily refined data interval.

For the eight catchments studied, none of the superposed events had the desired property of more than 50% dominance when considered at 1 hour data interval. As a result, application of the Jacobi process failed to converge on any of the eight catchments. In contrast the Gauss-Seidel process successfully produced an average unit hydrograph for each catchment. Performance of the Jacobi and Gauss-Seidel iterative methods is summarised in Table 6.2 for the eight superposed events.

Average unit hydrographs derived by event superposition and Gauss-Seidel iteration compared closely with those derived by event superposition and least-squares. The comparison is shown in Figure 6.1 for catchment 61001; the difference between derived unit hydrographs was scarcely noticeable on any of the eight catchments. An insight into how the two methods (as opposed to their solutions) differ is provided by Figure 6.2. This compares the superposed event quick response hydrograph with that generated by re-convolving the superposed net rainfall hyetograph with the unit hydrograph derived by iteration. Whereas the least-squares criterion ensures that all the informaton in the response hydrograph is used, Figure 6.2 confirms that, in the iterative method, only the central part of the response hydrograph is fitted to, the extent being determined by the range of influence of the dominant block of net rainfall.

TABLE 6.2 CONVERGENCE RESULTS FOR UNIT HYDROGRAPH DERIVATION BY JACOBI AND GAUSS-SEIDEL ITERATION (SUPERPOSED EVENTS)

Catchment	Dominance of peak net rainfall block	Number of iterations required for convergence* by method stated		
	%	Gauss-Seidel	Jacobi	Jacobi with under-relaxation†
23002	42.8	5)		13
45004	31.1	8)		13
46003	33.1	8)		16
53005	43.3	9)	DIVERGED	8
58001	27.0	13)		20
61001	41.1	8)		10
65001	18.5	14)		DIVERGED
76014	30.8	9)		>20

* *Convergence deemed when none of unit hydrograph ordinates changed by more than 5×10^{-5} between iterations (ordinates sum to near unity).*

† *See Section 6.5.*

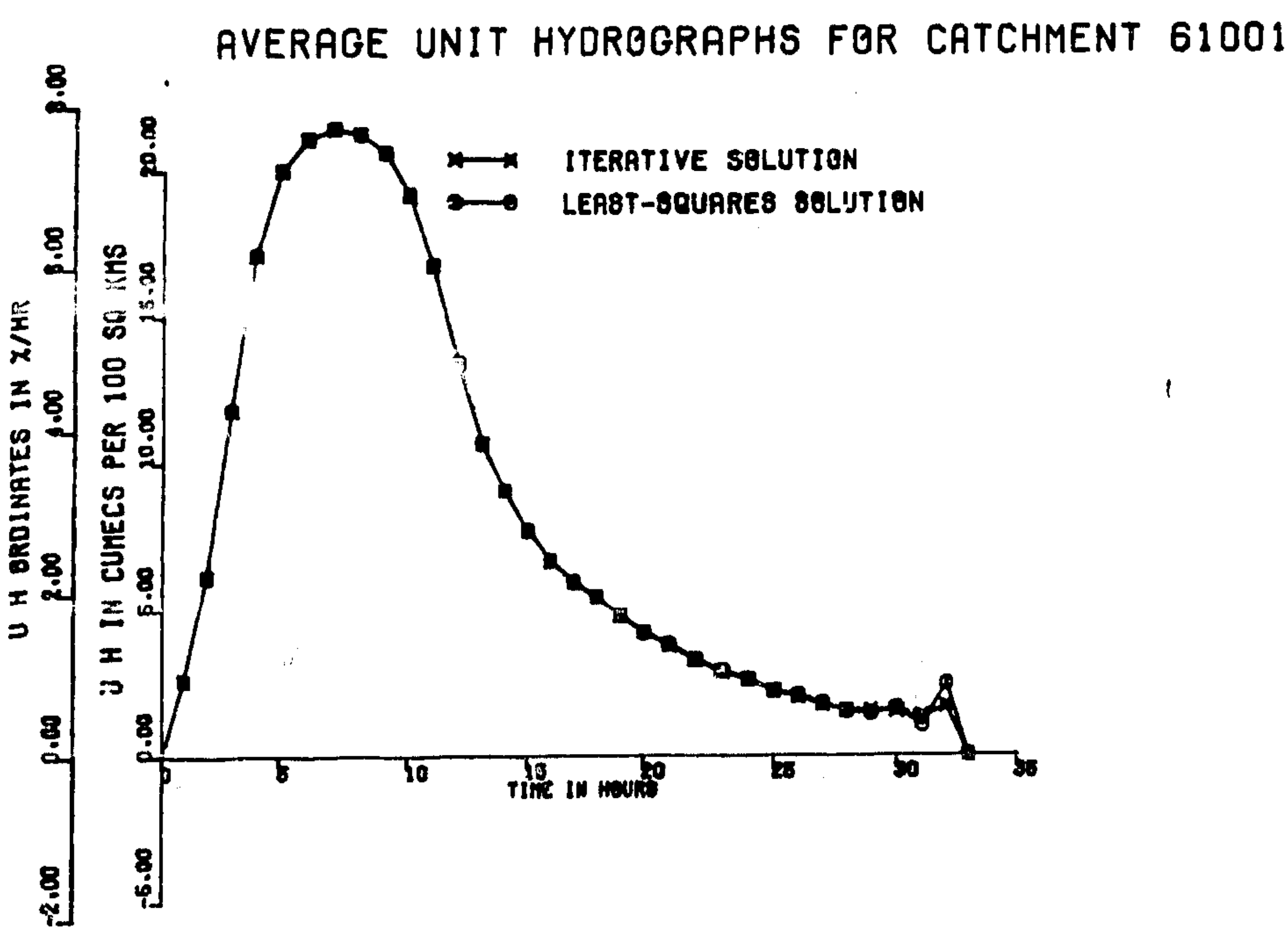
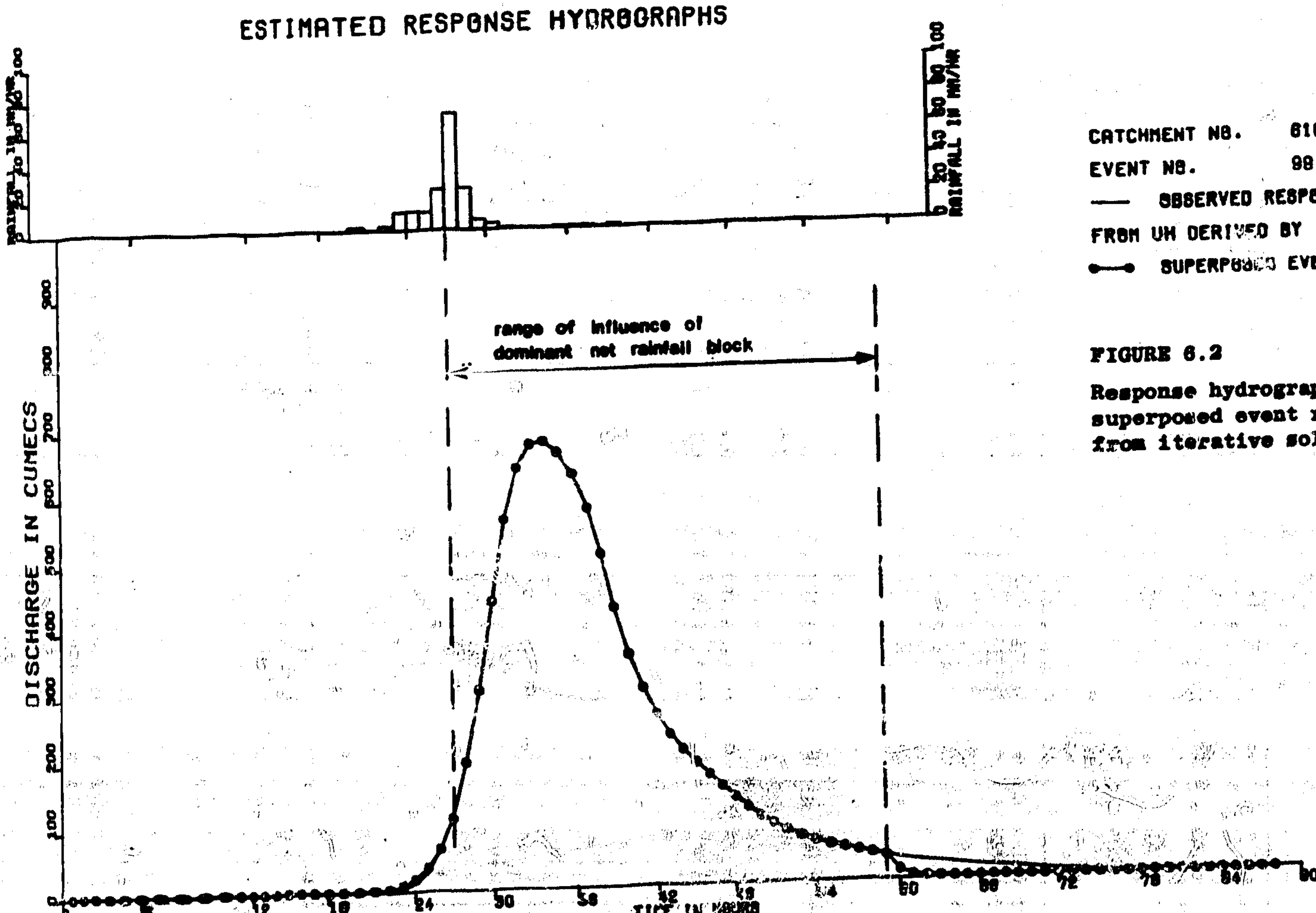


FIGURE 6.1 Comparison of least-squares and iterative solutions for superposed event



CATCHMENT NO. 61001

EVENT NO. 98

—— OBSERVED RESPONSE HYDROGRAPH

FROM UH DERIVED BY

●—● SUPERPOSED EVENTS GAUSS-SEIDEL ITERATION

FIGURE 6.2

Response hydrograph for superposed event reconstruction from iterative solution

6.4 Choice of initial estimate

A necessary ingredient of an iterative approach to solution of Equations 6.2 is an initial estimate of the unit hydrograph ordinates. Choice of a good initial approximation will generally reduce the number of iterations required for convergence. In the trials reported in Sections 6.2 and 6.3, an initial estimate of the unit hydrograph was made by scaling the relevant part of the quick response hydrograph to unit volume. In the notation of Equations 6.2:

$$u_i^{(o)} = y_{p+i+1} \Big/ \sum_{j=p}^{p+n-1} y_j$$

where $u_i^{(o)}$ denotes the initial estimate of u_j.

6.5 Potential for simple implementation

Combination of the event superposition concept and an iterative solution technique brings derivation of an average unit hydrograph within the scope of hand calculation. For such implementation it is a little easier if the unit hydrograph derivation is laid out in Jacobi format, the unknowns being updated simultaneously at the end of each iteration rather than successively during the iteration. However, the relative ease of using Jacobi iteration is offset by its general tendency toward slower convergence than Gauss-Seidel iteration, or indeed divergence. No categoric rules can be given but it is often possible to enhance the convergence performance of the Jacobi process by the technique of 'under-relaxation'. One version of this technique is to replace $\underline{u}^{(n)}$ by the average of $\underline{u}^{(n)}$ and $\underline{u}^{(n-1)}$ when iterating for $\underline{u}^{(n+1)}$. As can be judged from the results presented in the extreme right hand column of Table 6.2, this expedient was successful in correcting divergence of the Jacobi process on six of the eight catchments considered.

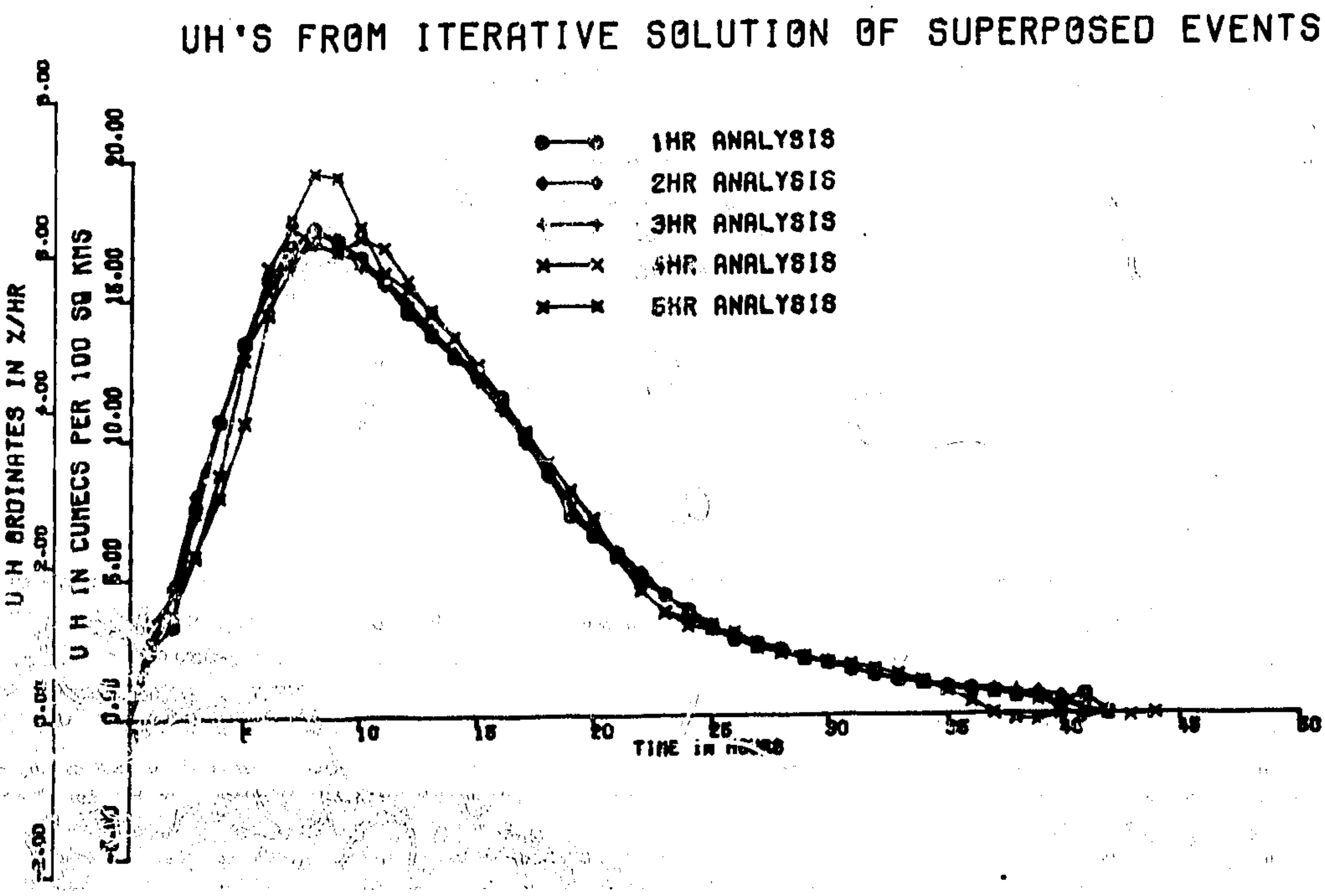


FIGURE 6.3 Effect of time interval used in analysis on 1-hour unit hydrographs

But to keep the solution process at its simplest - which is the essence of a ... calculation method - there is much to be said for increasing the interval at which the data are analysed until the superposed rainfall event attains the desired property of peak net rainfall block dominance. Not only does this strategy ensure convergence of the Jacobi process but at the same time it reduces the length of calculations to be carried out at each iteration. In the context of deriving an average unit hydrograph, the significance of the information loss resulting from use of a coarser data interval can be surprisingly small. As an example, Figure 6.3 shows average unit hydrographs derived for catchment 45004 by event superposition and Gauss-Seidel iteration, using data intervals ranging from 1 to 5 hours. It should be noted that the response functions plotted are all 1-hour unit hydrographs, the necessary conversions having been made using the familiar s-curve technique (eg Wilson (1974)). As can be seen from Figure 6.3 there was very little difference between the average unit hydrograph derived by 1 hour analysis and those derived using 2 and 3 hour data intervals. That resulting from the 4-hour analysis has an unsightly double peak (created by the interpolation algorithm used in the s-curve technique) but otherwise provides an acceptable approximation to the 1-hour analysis - even though the time to peak of the unit hydrograph is only twice the data interval (ie 6 hours). The effect of the choice of data interval on the dominance of the peak net rainfall block, and hence on the rate of convergence of the iterative processes, is illustrated in Table 6.3.

TABLE 6.3 EFFECT OF DATA INTERVAL ON PERFORMANCE OF JACOBI AND GAUSS-SEIDEL PROCESSES (SUPERPOSED EVENT ON CATCHMENT 45004)

Data interval	Dominance of peak net rainfall block	Number of iterations required for convergence of stated method	
hr	%	Jacobi	Gauss-Seidel
1	31.1	Diverged	8
2	48.9	>20	6
3	60.5	10	5
4	66.5	8	4
5	72.1	6	5

6.6 When further events are available for analysis

A final advantage of this simple method of deriving an average unit hydrograph - namely, event superposition followed by an iterative solution technique - is that additional events can be incorporated with particular ease. A new event is simply added to the existing superposed event and the iterative solution re-started using the previously derived average unit hydrograph as the initial estimate of the new solution.

7 CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

7.1 Conclusions

(i) There are many ways of determining a catchment average unit hydrograph. The report highlights some strengths and weaknesses of particular methods but, because the study was carried out using real rather than synthetic data, it was not possible to pronounce judgement on a best method. In truth there is no universally correct method because there is no exclusively valid definition of an average unit hydrograph.

(ii) Three techniques of unit hydrograph derivation were considered: the least-squares ordinate method and two variants thereof. It was confirmed that the basic least-squares method frequently produces a perturbed unit hydrograph when applied to individual events. Such perturbations are reduced or eliminated in the Flood Studies Report method, which incorporates a fixed amount of post-derivation smoothing. The effectiveness of the smoothing is dependent on the relative frequency of the oscillations in comparison to the data interval. An unwelcome feature of this type of smoothing is the tendency to reduce the magnitude of true variations between ordinates. The third method of derivation considered, the restricted least-squares technique, avoids this drawback and provides an effective, if complicated, way of deriving a smooth unit hydrograph from an individual event. However, it is not suitable for use on catchments where the characteristic response is bimodal.

(iii) Four ordinate by ordinate methods of averaging individual unit hydrographs were examined. Of these the median peaks aligned technique was found to reproduce most nearly results obtained by eye. The technique of shape factor averaging offers a practical alternative to these methods but lacks total objectivity.

(iv) Of the two joint analysis techniques considered, the newer one of event superposition produced average unit hydrographs in close agreement with those derived by event concatenation but had the advantage of requiring far less computation.

(v) In the context of deriving an average unit hydrograph from many events, the straightforward least-squares ordinate method generally proved adequate. Problems of instability that frequently feature in unit hydrographs derived from individual events had little influence on average unit hydrographs. With regard to methods of averaging, it was found that the median peaks aligned and event superposition techniques generally yield distinct results. The former tends to produce an average unit hydrograph that is rather too peaky whereas event superposition tends to produce one that is rather too diffuse. However, these differences are unlikely to be significant in terms of affecting the overall confidence with which a unit hydrograph based rainfall/runoff model can be applied.

(vi) For applications requiring no more than an average unit hydrograph, event superposition followed by least-squares derivation provides a particularly effective solution technique. A variant that replaces the least-squares criterion by an exact solution of the dominant equations is recommended as a powerful technique capable of manual implementation.

7.2 Recommendations for further study

(i) The interaction of data separation methods with th... ...on of an average unit hydrograph deserves closer attention. One possibility is to link the rainfall and runoff separation processes so that net rainfall and response runoff events are defined in a manner consistent with a fixed time base for the unit hydrograph. An extension of the approach might be to use a previously derived average unit hydrograph (together with a fully defined rule for runoff separation) to work backwards toward net rainfall, with the aim of resolving a method of rainfall separation in keeping with the adopted unit hydrograph. We hope to evaluate the feasibility and usefulness of this technique in future research.

(ii) A point not dealt with in the report is: how meaningful is an average unit hydrograph? A study that would go much of the way to answering this question would be to examine the sensitivity of an average unit hydrograph to the number of events used in its construction.

ACKNOWLEDGEMENT

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APPENDIX 1: RESTRICTED LEAST-SQUARES METHOD OF UNIT HYDROGRAPH DERIVATION

Note: The description is taken from Section 4.7 of Reed's thesis with minor amendment and re-ordering of material.

A1.1 Formulation

In the restricted least-squares method the unit hydrograph is represented by a subset of the original (ie evenly spaced) ordinates. To simplify the description the original ordinate positions will be referred to as grid points (see Figure A1.1(a)). The restricted formulation (Figure A1.1(b)) is in terms of unit hydrograph ordinates located at active grid points; those ordinates located at passive grid points are specified by linear interpolation between adjacent active ordinates. Thus the restricted unit hydrograph is made up of a number of straight line segments (Figure A1.1(b)).

The discrete convolution relation (see Section 2.1):

$$X\underline{u} = \underline{y} \qquad \text{A1.1}$$

requires modification to take account of the reduced formulation. If $\underline{u}'$ denotes the $n' \times 1$ vector of active ordinates then the interpolative relations define an $n \times n'$ transformation matrix, F, such that:

$$\underline{u} = F\underline{u}' \qquad \text{A1.2}$$

For illustrative purposes the transformation corresponding to the selection of active grid points depicted in Figure A1.1(b) is given in Figure A1.1(c). Substitution of Equation A1.2 into Equation A1.1 gives:

$$X'\underline{u}' = \underline{y} \qquad \text{A1.3}$$

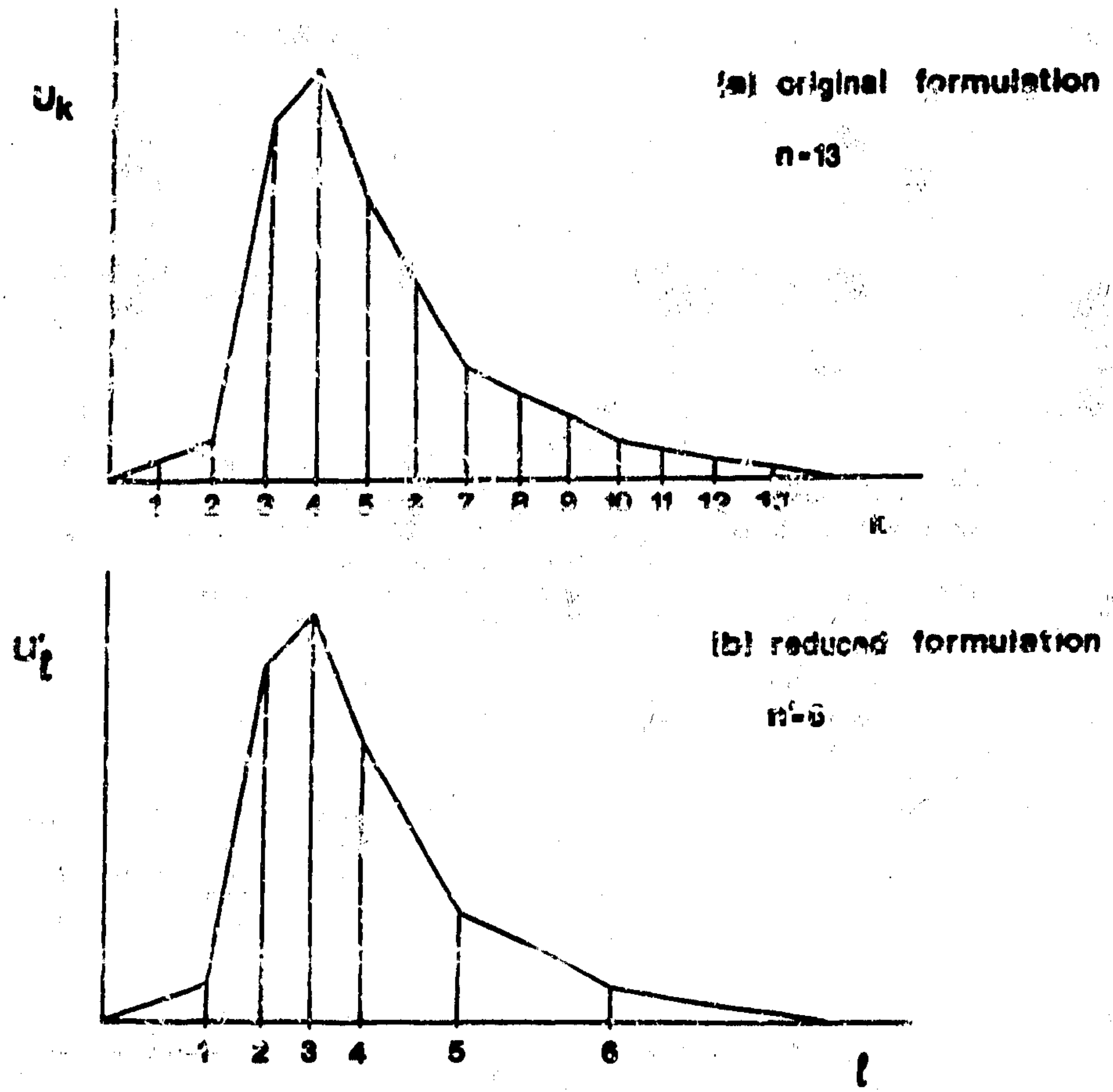
where $X' = XF$ is the modified coefficient matrix and is $m \times n'$. Equation A1.3 is solved by least-squares to yield $\hat{\underline{u}}'$, the segmented unit hydrograph.

A1.2 Purpose

The restricted approach is useful when the least-squares method (Section 2.2) produces an unsatisfactory unit hydrograph. By defining sufficient interpolations (ie passive grid points), irregularities in the derived unit hydrograph can be eliminated. It is interesting to note that unequal spacing of ordinates, placing emphasis on good representation of the rising limb and crest of the unit hydrograph, was a prominent feature of W M Snyder's original least-squares unit hydrograph method (Snyder 1955). Subsequently, Newton and Vinyard (1967) indicated the utility of a segmented representation for the avoidance of erratic unit hydrograph ordinates; however, an objective method for the selection of a suitable set of active grid points was not given. Before presenting such an algorithm it is necessary to define features that make a unit hydrograph acceptable or satisfactory.

A1.3 What is a satisfactory unit hydrograph?

Certain a *priori* information is available relating to the unit hydrograph. A characteristic property of the response runoff system is that sudden fluctuations



$$\underline{U} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \underline{U}'$$

(c) $\underline{U} = \underline{F}\underline{U}'$ for illustrated example

FIGURE A1.1 Example of formulation used in restricted least-squares method in net rainfall result in more restrained variation in response runoff; in other words the system is heavily damped. It is convenient to focus attention initially on the instantaneous unit hydrograph, $h(t)$; this represents the response of the system to the most acute input fluctuation, namely an instantaneous unit impulse.

Properties of $h(t)$ in accordance with it being the impulse response of a heavily damped, linear, time-invariant, non-anticipating and initially relaxed system include:

(i) $h(t) = 0$ for $t \leq 0$
(ii) $h(t) \geq 0$ for $0 < t < \infty$
(iii) $h(t) \to 0$ as $t \to \infty$
(iv) $\int_0^\infty h(t)dt = 1$

... catchments it is reasonable to expect also that the instantaneous unit hydrograph ... be unimodal and possess no more than one inflexion point on each limb,

$\frac{dh}{dt} \geq 0$ for $0 \leq t \leq tp$

$\frac{dh}{dt} \leq 0$ for $tp \leq t < \infty$

and (vi) $\frac{d^2h}{dt^2} \geq 0$ for $0 \leq t \leq t_1$ and $t_2 \leq t < \infty$

$\frac{d^2h}{dt^2} \leq 0$ for $t_1 \leq t \leq t_2$

where tp denotes the location of the peak value and t_1 and t_2 refer to points of inflexion on the rising and receding limbs. A hypothetical instantaneous unit hydrograph illustrating these features is shown in Figure A1.2.

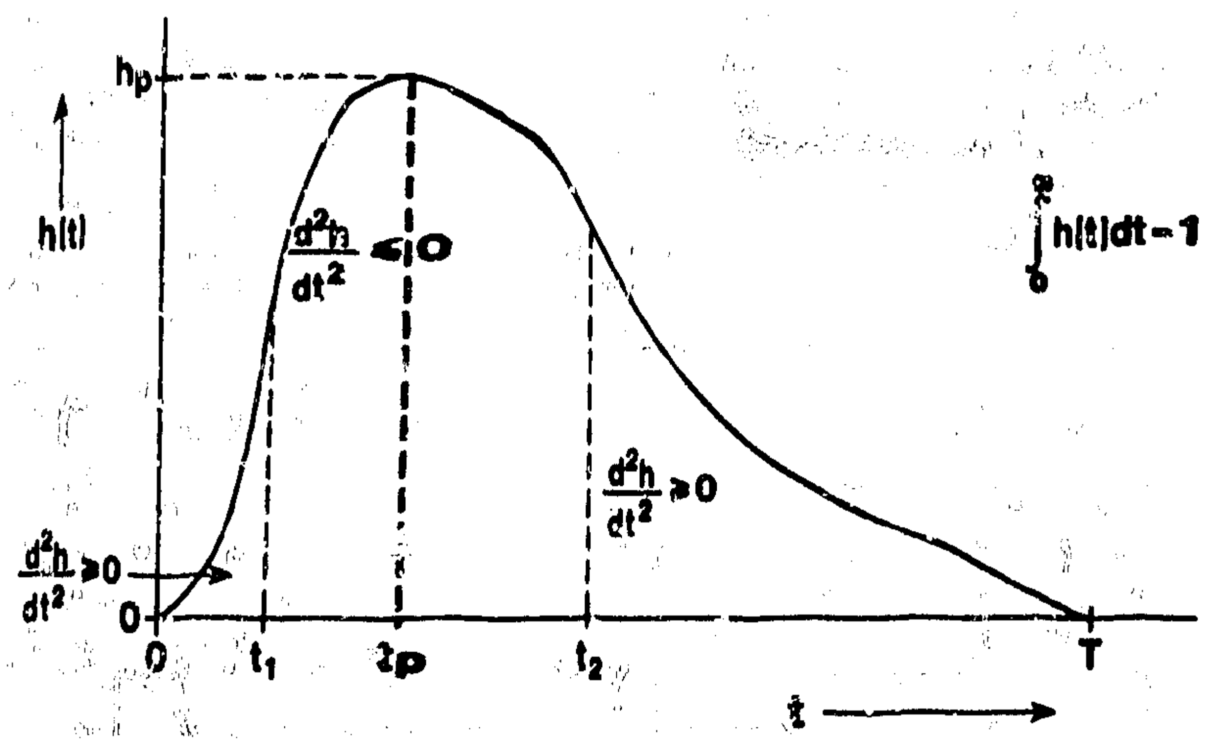


FIGURE A1.2 Desirable properties of an instantaneous unit hydrograph

The above properties apply in essence to the unit hydrograph also, the variable $h(t)$ being replaced by the ΔT-period unit hydrograph, $U(\Delta T,t)$. When used in conjunction with conventional rainfall and runoff separation techniques, a finite upper limit is placed on t, namely the time base, T, of the unit hydrograph. Applying the usual discretisation at equal intervals, the above properties can be interpreted in terms of the unit hydrograph ordinates, u_k, as follows:

(i) $u_k = 0$ for $k \le 0$

(ii) $u_k \ge 0$ for $1 \le k \le n$ where $(n+1)\Delta T = T$

(iii) $u_k = 0$ for $k > n$

(iv) $\sum_{k=1}^{n} u_k = 1$

(v) $u_k \ge u_{k-1}$ for $k = 1, 2, \ldots, k_p$
$u_k \le u_{k-1}$ for $k = k_p+1, \ldots, n+1$

and

(vi) $g_k \ge g_{k-1}$ for $k = 1, \ldots, k_1$ and $k = k_2+1, \ldots, n+2$
$g_k \le g_{k-1}$ for $k = k_1+1, \ldots, k_2$

where k_p corresponds to the peak ordinate, $g_k \equiv u_k - u_{k-1}$, and k_1 and k_2 relate to the location of the inflexion segments on the rising and receding limbs.

A unit hydrograph that satisfies the above properties, with the exception of (iv), will be termed <u>satisfactory</u>. Property (iv) is excluded because the volume constraint is not particularly easy to implement in the restricted formulation. The deviation of the volume from unity provides one of several indications of the acceptability of the unit hydrograph (see Section A1.5).

A1.4 Algorithm for derivation of a satisfactory unit hydrograph

STEP 1 Initially all ordinates are assumed active; thus $F = I_n$, the n x n identity matrix, in Equation A1.2.

STEP 2 The matrix X' is formed (X' = XF) and Equation A1.3 solved by least-squares to yield the segmented unit hydrograph, $\underline{\hat{u}}'$.

STEP 3 The segmented unit hydrograph (from Step 2) is examined as follows:

(i) The peak ordinate is located and the gradient of each line segment computed.

(ii) If the gradient of the first line segment is negative then the first active ordinate is selected for elimination. Similarly if the gradient of the last line segment is positive then the last active ordinate is selected for elimination.

(iii) The gradient of each line segment on the rising limb, excluding the first and last such segments, is examined to see if it constitutes a local minimum gradient. Similarly the gradient of each line segment on the receding limb, excluding the first and last such segments, is examined to see if it constitutes a local maximum gradient. Line segments that fall into either of these categories transgress the inflexion segment criterion (property (vi)) of a satisfactory unit hydrograph. The ordinates at each end of an offending segment are selected for elimination, to be replaced by a single re-activated ordinate. A weighted-average formula based on gradient differences selects for re-activation that ordinate which is likely to be most effective in reducing the jaggedness of the representation. Sometimes (for example, when there is no intervening grid point) the ordinate selected for re-activation will be one of the two just selected for elimination.

STEP 4 The matrix F is modified, and the vector of unknowns reduced in length, to take account of the eliminations and replacements selected in Step 3. If no eliminations have been selected then $\hat{\underline{u}}'$ is the required satisfactory segmented unit hydrograph. Otherwise the procedure is re-entered at Step 2.

An example of how the algorithm progressively restricts the unit hydrograph representation, until a satisfactory solution is derived, is captured in Figure A1.3 for [illegible]t 4 on catchment 45004.

A1.5 Acceptability of the segmented unit hydrograph

While the unit hydrograph derived by the restricted least-squares method may be satisfactory in the sense defined in Section A1.3, its actual acceptability can be judged in a number of ways. Firstly, if the derived unit hydrograph comprises fewer than (say) six line segments then it is an indication that the algorithm has had quite a struggle to arrive at a satisfactory unit hydrograph. A second sign that all may not be well is if the volume of the derived unit hydrograph deviates significantly from unity. Perhaps the clearest check on the implications of using the algorithm is a visual comparison of the unit hydrograph derived by restricted least-squares with that derived by the basic least-squares method (Step 1 of the algorithm).

A1.6 References

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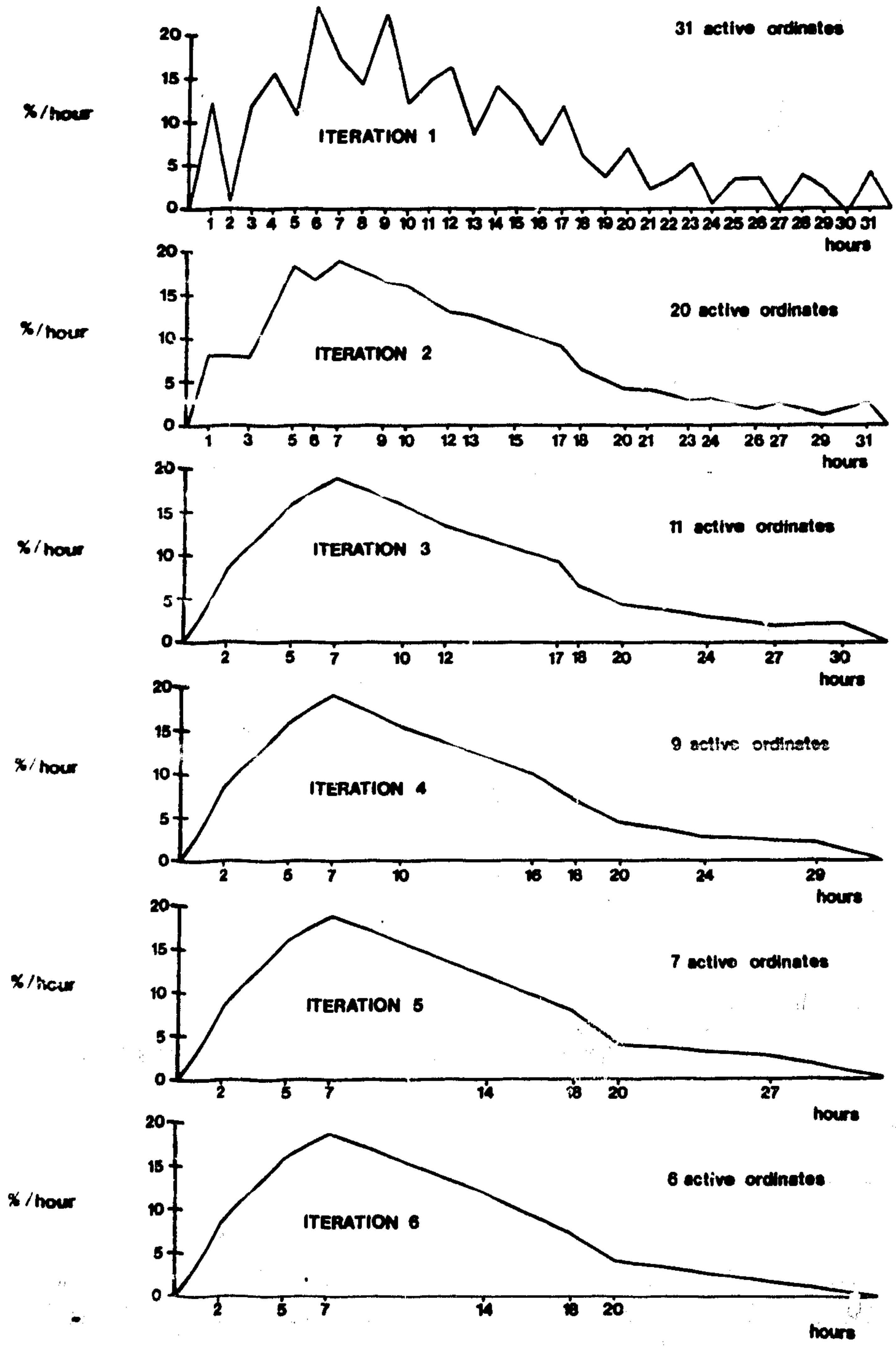

FIGURE A1.3 Progressive restriction to derive an acceptable unit hydrograph for event 4 on catchment 45004

APPENDIX 2: CHARACTERIZATION BY SHAPE FACTORS

A2.1 Moment integrals

volume: $V = \int_0^\infty f(t)dt$

(where $f(t)$, $0 \leq t < \infty$ is the distribution to be characterized).

mean: $M_1' = \int_0^\infty tf(t)dt \Big/ \int_0^\infty f(t)dt$

m'th moment about mean:

$$M_m = \int_0^\infty (t-M_1')^m f(t)dt \Big/ \int_0^\infty f(t)dt$$

eg variance:

$$M_2 = \int_0^\infty (t-M_1')^2 f(t)dt \Big/ \int_0^\infty f(t)dt$$

A2.2 Shape coefficients

coefficient of variation : $c_2 = \sqrt{M_2}/M_1'$

coefficient of skewness : $c_3 = M_3/M_2^{3/2}$

coefficient of peakedness : $c_4 = (M_4/M_2^2) - 3$

Notes: *(i)* $c_3 > 0$ *implies distribution is skewed to the left.*

(ii) $c_4 > 0$ *implies that distribution is more peaky than the normal distribution (given the same mean and variance).*

A2.3 Shape factors

It is convenient to refer to the volume, the mean, and the coefficients of variation, skewness, and peakedness collectively as shape factors. Strictly speaking, the volume and mean characterize the scale and location of the distribution rather than its shape.