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A STUDY OF STEADY-STATE FLOW TO PARTIALLY-PENETRATING WELLS

by

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SUMMARY

A formula is obtained for the extra head loss due to partial penetration of a well under steady-state flow conditions. The formula, which is a simplification of a previously developed approximation, is of adequate accuracy for penetrations greater than about 20% yet is readily evaluated on a calculator or in a computer spreadsheet.

In the development, it is assumed that inflow is uniformly distributed along the screen of the well; this simple assumption is shown to be superior to a recently advocated distribution.

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1 INTRODUCTION

The British Geological Survey has recently completed a comprehensive study of irrigation well design for Bangladesh. As one element of the study (Barker and Herbert, 1989), a set of formulae were presented for the estimation of well losses: the formulae had to be sufficiently simple to be incorporated in a spreadsheet for comparing the costs of various well designs.

In the study area in Bangladesh, the wells are almost always fully-penetrating, so the issue of partial penetration did not arise. However, to make the results of that study of wider applicability a brief study of partial-penetration was undertaken and is reported here.

There have been several studies of partial penetration (see, for example, Hantush (1964), Anon (1964) and Sternberg (1973), and references therein). The purpose of the study was neither to review previous work nor to develop any fundamentally new approach. Rather the intention was to obtain a very simple formula for approximating the head loss due to partial penetration.

An existing set of approximate formulae (Appendix A) were further simplified and compared with a relatively accurate equation (derived in Appendix B). These *accurate* equations are based on a model taking the form of a finite-diameter well which partially penetrates a confined homogeneous anisotropic aquifer, with uniformly distributed inflow along the screen. This model is consistent with the model used in the Bangladesh study.

The uniform inflow boundary condition on the screen is arguably less realistic than a constant-head condition (although the latter condition is not exact because of well losses). In Appendix C consideration is given to the uniformity of the head distribution that results both from the uniform inflow and from a distribution of inflow advocated by Haitjema and Kraemer (1988).

The possible presence of a gravel pack will not be considered. However, it is often the case that the outer surface of the gravel pack is the surface to which the flow converges due to partial penetration, and should thus replace the well radius in the following analysis. (Head losses through the gravel pack will usually be adequately estimated by the Thiem equation, or rather its generalization mentioned below.)

2 DRAWDOWN DUE TO PARTIAL PENETRATION

Consider the steady-state flow system shown in Figure 1. The average drawdown over the whole aquifer thickness at the well radius is given by

$$\bar{s}_w = \frac{1}{b}\int_0^b s(r_w, z)\,dz \qquad (1)$$

Taking the drawdown at the outer boundary of the aquifer to be zero (arbitrarily but without loss of generality), this average drawdown is given by

$$\bar{s}_w = \frac{Q_w}{2\pi b K_h}\ln\frac{R}{r_w} \qquad (2)$$

where b is the full aquifer thickness and K_h is the horizontal hydraulic conductivity of the aquifer (see Figure 1 and Section 5 for notation).

Equation (2) represents a generalization of the well-known Thiem equation, a proof of which is given in Barker and Herbert (1989, Appendix A).

The average drawdown over the well screen is

$$\bar{s}_s = \frac{1}{b_w - b_c}\int_{b_w}^{b_c} s(r_w, z)\,dz \qquad (3)$$

which depends on the parameters b_c , b_u and K_v in addition to those appearing in (2).

The difference between the two average drawdowns: $\bar{s}_w$ (over the full thickness of the aquifer at the radius of the screen) and $\bar{s}_s$ (over the penetration depth of the screen) will be referred to as the *penetration loss*.

The use of vertically averaged drawdowns is necessitated by the variation in drawdown along the well screen which results from the uniform-inflow assumption.

Some authors represent the additional drawdown due to partial penetration by a term of the form $2s_pQ_w/4\pi T$, so the quantity s_p (sometimes referred to as a *pseudo-skin factor*) is

$$s_p = \frac{2\pi T}{Q_w}(\bar{s}_s - \bar{s}_w) \tag{4}$$

where T $(=bK_h)$ is the transmissivity.

2.1 Exact solution

The exact solution for the model depicted in Figure 1 (which is derived in Appendix B) is

$$\frac{2\pi b K_h}{Q_w}(\bar{s}_s - \bar{s}_w) = \sum_{n=1}^{\infty} C_n(\kappa) \tag{5}$$

where

$$C_n(\kappa) = \left[\frac{I_0(n\kappa)K_0(n\kappa_w) - K_0(n\kappa)I_0(n\kappa_w)}{I_1(n\kappa_w)K_0(n\kappa) + K_1(n\kappa_w)I_0(n\kappa)}\right]\frac{2(\sin n\zeta_w - \sin n\zeta_c)^2}{n^3\kappa_w(\zeta_w - \zeta_c)^2} \tag{6}$$

$$\zeta_w = \pi b_w / b \tag{7}$$

$$\zeta_c = \pi b_c / b \tag{8}$$

$$\kappa = \frac{\pi R}{b}\sqrt{\frac{K_v}{K_h}} \tag{9}$$

and

$$\kappa_w = \frac{\pi r_w}{b}\sqrt{\frac{K_v}{K_h}} \tag{10}$$

The dependence of C_n on κ and hence on the aquifer radius, R, is emphasized because, in general, it is not possible to assign a value to this radius. Fortunately, this problem is overcome by noting that

$$\lim_{x \to 0} \frac{K_0(x)}{I_0(x)} = 0 \tag{11}$$

The coefficient $C_n(R)$ therefore converges to a finite limit as R tends to infinity:

$$C_n(\infty) = \left[\frac{K_0(n\kappa_w)}{K_1(n\kappa_w)}\right] \frac{2(\sin n\zeta_w - \sin n\zeta_c)^2}{n^3 \kappa_u (\zeta_u - \zeta_c)^2} \tag{12}$$

Equation (5) with (12) gives the asymptotic penetration loss in the steady state as the radius of influence increases. The speed of convergences is demonstrated in Figure 2 where $\sum C_n(\kappa)/\sum C_n(\infty)$ is plotted against κ, for $\kappa_w = 10^{-3}$ and a centrally placed screen. This result indicates that the aquifer radius need only be about the same as the aquifer thickness ($\kappa = 1$ for and isotropic aquifer) for (12) to adequately approximate (6); and it will therefore be appropriate after any reasonable period of steady pumping of a well.

2.2 Simple approximate solution

Although the solution given by (5) and (12) is of practical value in a scientific computing environment, it is far too complex for hand calculation or to be incorporated effectively into a spreadsheet program. Also, the series is slowly convergent for certain combinations of parameter values.

It was therefore considered necessary to develop a simpler approximate solution. Anon (1964) gives an estimate (based on flow to a line source) of the extra head loss due to partial penetration in an isotropic aquifer (see Appendix A). Following Huisman (1972), we can write the result in the form:

$$\Delta s_p = \frac{Q_w}{2\pi b K_h} \frac{(1-p)}{p} \ln \frac{\pi \theta p}{\kappa_w} \tag{13}$$

where $p=(b_w-b_c)/b$ is the fractional penetration, and θ is a tabulated function of the degree of penetration and the eccentricity of the screen with respect to the centre of the aquifer. For a well which extends from the bottom (or top) of the aquifer, Huisman (1972) suggests that the following approximation is valid for penetrations greater than 20%:

$$\theta = 1 - p \tag{14}$$

If the screen is centrally positioned between the top and the bottom of the aquifer, the previous two formulae apply to both the upper and lower half of the aquifer (by symmetry). Factors of 1/2 in the discharge rate and aquifer thickness will cancel so (13) still applies to the whole aquifer provided:

$$\theta = \frac{1-p}{2} \tag{15}$$

Although Huisman provides a table of θ values for intermediate degrees of eccentricity of the screen, interpolation on the table is not convenient. The approximation has therefore been developed further giving an entirely analytic formula.

First an eccentricity parameter, η , is introduced (which is different from that, e, of Huisman, 1972):

$$\eta = \frac{2e}{1-p} \tag{16}$$

where

$$e = \frac{z_c}{b} \tag{17}$$

and z_c is the vertical displacement of the centre of the screen from the centre of the aquifer. The value of η ranges from zero (for a centrally placed screen) to unity (for a screen extending from the top or base of the aquifer). Equations (15) and (16) suggest the introduction of an estimate, $\tilde{\theta}(p, \eta)$, of θ which has the properties

$$\tilde{\theta}(p, 0) = (1 - p)/2 \tag{18}$$

and

$$\tilde{\theta}(p, 1) = (1 - p) \tag{19}$$

From the equations given in Appendix A it can be deduced that the derivative of θ with respect to eccentricity is zero when the eccentricity is zero:

$$\frac{\partial\tilde{\theta}}{\partial\eta}(p, 0) = 0 \tag{20}$$

This condition suggests the use of a quadratic functional form. Both of the following forms satisfy all three conditions (18)-(20)

$$\tilde{\theta}(p, \eta) = \frac{(1 - p)}{(2 - \eta^2)} \tag{21}$$

$$\tilde{\theta}(p, \eta) = (1 - p)(1 + \eta^2)/2 \tag{22}$$

2.3 Comparison of exact and approximate solutions

The accuracy of the approximation expressed by (13) with (21) or (22) has been investigated by comparison with results from (5) for an infinite aquifer. Figures 3 and 4 show the percentage error in Δs_p as a function of the eccentricity, η:

$$\frac{\bar{s}_s - \bar{s}_u - \Delta s_p}{\bar{s}_s - \bar{s}_u} \times 100 \qquad (23)$$

The errors are all acceptably small, and there is little to choose between the accuracy of (21) and (22). Equation (21) is preferred slightly because it is less biassed overall; although (22) is clearly better at moderate eccentricities.

As the penetration increases the penetration loss becomes small in relation to other losses, so errors at larger penetrations are less important and equation (13) is sufficiently accurate for most practical purposes for all penetrations greater than about 20%.

3 DISCUSSION AND CONCLUSIONS

A previously developed formula for the extra head loss, Δs_p , due to partial penetration has been simplified further to make it easier to apply. The result is:

$$\Delta s_p = \frac{Q_w}{2\pi b K_h} \frac{(1-p)}{p} \ln\left(\frac{p(1-p)}{(2-\eta^2)} \frac{b}{r_w} \sqrt{\frac{K_h}{K_v}} \right) \tag{24}$$

where p is the fractional penetration of the well screen and η is the eccentricity of the screen with respect to the centre of the aquifer (see (16) and (17)). The formula is considered adequate for penetrations greater than about 20%.

Any estimation of the extra drawdown due to partial penetration will be subject to several unavoidable errors, which include:-

(a) Aquifers are all heterogeneous to some extent.

(b) It is often difficult to estimate the full aquifer thickness.

(c) The formation close to the well is often damaged as a result of drilling.

In view of these it is concluded that, although accurate formulae are available, the above formula can be used in most practical situations.

The study was based on the assumption of uniform inflow along the well screen. Although this assumption will always be an approximation, is shown (Appendix C) to be more consistent with the alternative assumption of constant head along the screen than a non-uniform inflow distribution used by Haitjema and Kraemer (1988).

4 REFERENCES

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5 NOTATION

b	thickness of the aquifer.
b_c	vertical distance from the top of the aquifer to the top of the screen.
b_w	depth of the bottom of the screen below the top of the aquifer.
$C_n(\kappa)$	see (6).
e	$= z_c/b$
$H(x)$	function given by (A3).
I_o, I_1	modified Bessel functions of the first kind.
J_0	Bessel function.
K_h	the horizontal hydraulic conductivity of the aquifer.
K_v	the vertical hydraulic conductivity of the aquifer.
K_0, K_1	modified Bessel functions of the second kind.
l	half the screen length $(=(b_w - b_c)/2)$.
n	number of a term in a Fourier expansion, see (5).
p	fraction of the aquifer depth occupied by the screen, $=(b_w - b_c)/b$.
$q(z)$	inflow rate per unit length of the screen.
q_s	singular flux distribution, see (C11).
q_u	uniform inflow distribution, $=Q_w/2l$.
Q_w	total abstraction rate from the well.
r	radial distance from the well.
r_w	radius of the well screen.

R	outer radius of aquifer, at which the drawdown is zero.
$s(r,z)$	drawdown in the aquifer.
s_p	*pseudo skin factor*, see (4).
Δs_p	extra head due to partial penetration, see (13) and (24).
$\bar{s}_s$	average drawdown over the screened interval of a well, see (3).
$\bar{s}_w$	average drawdown over full aquifer thickness, see (1) and (2).
T	transmissivity of the aquifer, $=bK_h$.
z	vertical coordinate, see Figures (1) and (C1).
$\gamma(\zeta)$	dimensionless vertical distribution of flow through a screen.
$\gamma_c(\omega)$	cosine transform of $\gamma(\zeta)$.
ζ	$=\pi z/b$
ζ_w	$=\pi b_w/b$
ζ_c	$=\pi b_c/b$
η	eccentricity factor, see (16).
$\theta(p,e)$	function in Huisman's equation, (13).
$\tilde{\theta}(p,\eta)$	function, approximating Θ, in the Huisman equation.
κ	see (9).
κ_w	$=\kappa\rho_w$.
λ	see (C5).
ρ	$=r/R$.
ρ_w	$=r_w/R$.
$\sigma(\rho,\zeta)$	dimensionless drawdown.

$\sigma_c(\rho, n)$ cosine transform of σ.

$\bar{\sigma}$ average dimensionless drawdown over a vertical interval at a given radius, see (B26).

ω parameter of the half-line cosine transform.

APPENDIX A

Results from Anon (1964)

The following results are taken form Anon (1964, p82 & p95), but are presented in terms of the notation used in this report.

$$\Delta s_p = \frac{Q_w}{2\pi b K_h}\left(\frac{1-p}{p}\right)\left(\ln\frac{4b}{r_w}+\frac{1}{2}\ln\frac{K_h}{K_v}-F(p,e)\right) \qquad (A1)$$

where

$$F(p,e) = \frac{1}{p(1-p)}[2H\left(\frac{1}{2}\right)-2H\left(\frac{1}{2}-\frac{p}{2}\right)+2H(e)-H\left(e-\frac{p}{2}\right)-H\left(e+\frac{p}{2}\right) \qquad (A3)$$

and

$$H(x) = H(-x) = \int_0^x \ln\left[\frac{\Gamma\left(\frac{1}{2}-u\right)}{\Gamma\left(\frac{1}{2}+u\right)}\right]du \qquad (A3)$$

The quantity θ given in (13) is related to the function F by

$$\theta(p,e) = 4 \quad \exp[-F(p,e)] \qquad (A4)$$

APPENDIX B

Derivation of the general solution for partial penetration

Consider steady-state flow in a homogeneous, anisotropic, confined aquifer from a fixed-head outer boundary to a partially-penetrating well (Figure 2). Combining Darcy's law with the conservation equation gives

$$\frac{K_h}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + K_v\frac{\partial^2 s}{\partial z^2} = 0 \qquad r_w < r < R \ . \ 0 < z < b \qquad (B1)$$

At the outer boundary the drawdown is, without loss of generality, taken as zero:

$$s(R, z) = 0 \qquad (B2)$$

Since the aquifer is confined

$$\frac{\partial s}{\partial z}(r, 0) = \frac{\partial s}{\partial z}(r, b) = 0 \qquad (B3)$$

The flux along the well screen is given by Darcy's law while that along the well casing is zero:

$$2\pi r_w K_h \frac{\partial s}{\partial r}(r_w, z) = q(z) \qquad b_c < z < b_w$$

$$= 0 \qquad \text{otherwise} \qquad (B4)$$

Before solving these equations it is convenient to transform to a dimensionless form using

$$\zeta = \frac{\pi z}{b} \qquad (B5)$$

$$\rho = \frac{r}{b} \qquad (B6)$$

and

$$\sigma = \frac{2\pi K_h b s}{Q_w} \tag{B7}$$

Then (B1)-(B4) become

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\sigma}{\partial\rho}\right) + \kappa^2\frac{\partial^2\sigma}{\partial\zeta^2} = 0 \tag{B8}$$

where

$$\kappa = \frac{\pi R}{b}\sqrt{\frac{K_v}{K_h}} \tag{B9}$$

$$\sigma(1,\zeta) = 0 \tag{B10}$$

$$\frac{\partial\sigma}{\partial\zeta}(\rho,1) = \frac{\partial\sigma}{\partial\zeta}(\rho,\pi) = 0 \tag{B11}$$

$$\rho_w\frac{\partial\sigma}{\partial\rho}(\rho_w,\zeta) = -\psi(\zeta) \qquad \zeta_c < \zeta < \zeta_w$$

$$= 0 \qquad \text{otherwise} \tag{B12}$$

where

$$\psi(\zeta) = \frac{q(z)b}{Q_w} \tag{B13}$$

The solution of these equations can now be obtained with the help of the finite cosine transform:

$$f_c(n) = \int_0^\infty f(\zeta)\cos(n\zeta)d\zeta \tag{B14}$$

Equation (B8) transforms to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\sigma_c}{d\rho}\right)=n^2\kappa^2\sigma_c \qquad (B15)$$

which has the solution

$$\sigma_c(\rho,n)=A_nK_0(n\kappa\rho)+B_nI_0(n\kappa\rho) \qquad n>0 \qquad (B16)$$

$$\sigma_c(\rho,0)=A_0\ln\rho+B_0 \qquad n=0 \qquad (B17)$$

Using (B10), the outer boundary condition, (B16) and (B17) give

$$A_nK_0(n\kappa)+B_nI_0(n\kappa)=0 \qquad n>0 \qquad (B18)$$

and

$$B_0=0 \qquad (B19)$$

respectively.

The transformation of (B12) is

$$\rho_w\frac{d\sigma_c}{d\rho}(\rho_w,n)=-\psi_c(n) \qquad (B20)$$

and, using (B16) and (B17), this gives

$$A_nK_1(n\kappa\rho_w)-B_nI_1(n\kappa\rho_w)=\frac{\psi_c(n)}{\rho_w n\kappa} \qquad n>0 \qquad (B21)$$

and

$$A_0=-\psi_c(0)=-\pi \qquad (B22)$$

Solving (B18) and (B21) for the unknown coefficients A_n and B_n and substituting back into (B16) gives

$$\sigma_c(\rho,n) = \left[\frac{I_0(n\kappa)K_0(n\kappa\rho) - K_0(n\kappa)I_0(n\kappa\rho)}{K_0(n\kappa)I_1(n\kappa\rho_w) + K_1(n\kappa\rho_w)I_0(n\kappa)}\right]\frac{\psi_c(n)}{n\kappa\rho_w} \qquad n>0 \qquad (B23)$$

Similarly, B(17) becomes

$$\sigma_c(\rho,0) = -\pi \ln \rho \qquad (B24)$$

Using the above two equations, the general solution for the drawdown at any point in the aquifer can be written in the form of the inverse cosine transform:

$$\sigma(\rho,\zeta) = \frac{\sigma_c(\rho,0)}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\sigma_c(\rho,n)\cos(n\zeta) \qquad (B25)$$

The quantity of interest is the difference between the average drawdowns over the screen and the whole aquifer thickness. In general, the average drawdown between two depths z_a and z_b is given, in dimensionless form, by

$$\bar{\sigma}(\rho;\zeta_a,\zeta_b) = \frac{1}{\zeta_b - \zeta_a}\int_{\zeta_a}^{\zeta_b}\sigma(\rho,\zeta)d\zeta$$

$$= \frac{\sigma_c(\rho,0)}{\pi} + \frac{2}{\pi(\zeta_b - \zeta_a)}\sum_{n=1}^{\infty}\sigma_c(\rho_w,n)\frac{[\sin(n\zeta_b) - \sin(n\zeta_a)]}{n} \qquad (B26)$$

So the required difference is

$$\bar{\sigma}(\rho_w;\zeta_c,\zeta_w) - \bar{\sigma}(\rho_w;0,\pi) = \frac{2}{\pi(\zeta_w - \zeta_c)}\sum_{n=1}^{\infty}\sigma_c(\rho,n)\frac{[\sin(n\zeta_w) - \sin(n\zeta_c)]}{n} \qquad (B27)$$

The above formula can be applied once a specific distribution of flux into the well screen has been chosen. The simplest example is that of a uniform flux:

$$q(z) = \frac{Q_w}{b_s - b_c} \tag{B28}$$

This gives, using (B13) and (B14),

$$\psi_c(n) = \int_{\zeta_c}^{\zeta_w} \left(\frac{b}{b_w - b_c} \right) \cos(n\zeta) d\zeta = \frac{\sin(n\zeta_w) - \sin(n\zeta_c)}{n(\zeta_w - \zeta_c)} \tag{B29}$$

In summary, the dimensionless penetration loss for uniformly distributed inflow is given by (B27) in conjunction with (B23) and (B29).

APPENDIX C

Uniformity of the distribution of inflow

In this appendix consideration is given to the question of the validity of the uniform-inflow assumption.

The flow distribution into the screen will depend on the head variation within the screen, which in turn depends on the head losses within the screen. Even if there were no well losses, the distribution of flow into a partially-penetrating well would not be uniform: in general, there is bound to be some concentration of flow at the ends of the well screen.

The problem of finding the distribution of flow into a well screen, even for the relatively simple case of a constant drawdown along the screen, is (to the best knowledge of the author) as yet unsolved. However, it is not difficult to find a formula for the head-distribution corresponding to a given flow distribution. So the approach adopted here is to consider the uniformity of the head distribution corresponding to given inflow distributions, when there are no well losses.

The flow distribution into a screen will vary from uniform (assuming no well losses) when the well is fully penetrating, to most non-uniform when the well is in a vertically infinite aquifer. It is therefore appropriate to study this extreme latter case (Figure C1). The flow equations are the same as those given in Appendix B, but the outer boundary conditions are replaced by

$$\lim_{z^2+r^2\to\infty} s(r,z)=0 \qquad r\geq r_w \tag{C1}$$

The solution of the equations is effected in essentially the same way as in Appendix B, but with the half-line cosine transform replacing the finite transform. The dimensionless drawdown is found to be given by

$$\sigma(\rho,\zeta;\lambda)=\frac{4\pi K_h l}{Q_w}s(r,z)=\frac{2}{\pi}\int_0^\infty \frac{\dot{\gamma}_c(\omega)K_0(\omega\lambda\rho)\cos(\omega\zeta)}{\omega\lambda K_1(\omega\lambda)}d\omega \tag{C2}$$

where

$$\zeta = z/l \tag{C3}$$

$$\rho = r/r_w \tag{C4}$$

$$\lambda = \frac{r_w}{l}\sqrt{\frac{K_v}{K_h}} \tag{C5}$$

and the cosine transform of the flow distribution into the screen is given by

$$\gamma_c(\omega) = \int_0^\infty \gamma(\zeta)\cos(\omega\zeta)d\zeta \tag{C6}$$

where

$$\gamma(\zeta) = \frac{2lq(z)}{Q_w} \tag{C7}$$

The average drawdown along the screen is also of interest ((C13) below) and is given, in dimensionless form, by

$$\bar{\sigma}_w(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{\gamma_c(\omega)K_0(\omega\lambda\rho)\sin(\omega)}{\omega^2\lambda K_1(\omega\lambda)}d\omega \tag{C8}$$

For the case of a uniform flux

$$\begin{aligned} q_u(z) &= \frac{Q}{2l} \qquad |z| < l \\ &= 0 \qquad |z| > l \end{aligned} \tag{C9}$$

the transformed dimensionless flux is given by

$$\gamma_{uc}(\omega) = \int_0^1 \cos(\omega\zeta)d\zeta = \frac{\sin\omega}{\omega} \qquad (C10)$$

Another flux distribution worth consideration is that recently advocated by Haitjema and Kraemer (1988) which has the form

$$q_s(z) = \frac{Q_w}{\pi\sqrt{l^2 - z^2}} \qquad |z| < l$$

$$= 0 \qquad |z| > l \qquad (C11)$$

Note that this function is singular at the ends of the screen, but it has a finite integral, equal to Q_w, along the screen.

The distribution given by (C11) is exact for the two-dimensional problem of planar flow through a finite slot across which there is a uniform potential. Polubarinova-Kochina (1962) argued that the same distribution would become precise for the partial-penetration problem, with constant head within the screen, as the well radius tends to zero. Haitjema and Kraemer (1988) (following Muskat (1937)) point out, however, that in that limit ($r_w \to 0$) the inflow distribution should become uniform. What none of these researchers appeared to realize is that (C11) must be the limiting form of the distribution as the radius of the well tends to ***infinity***, since it is then that the cylindrical problem tends to the planar problem.

The dimensionless transform of $q_s(z)$ is given, from (C6), by

$$\gamma_{sc}(\omega) = J_0(\omega) \qquad (C12)$$

where $J_0(x)$ is a Bessel function.

Equations (C2) and (C8) give the drawdown distribution (and its average) along the screen. However, the problem of interest is that of the uniformity of this distribution for a given flux distribution. There are many possible ways of expressing this uniformity in quantitative terms.

For computational convenience the measure of uniformity used here is the percentage deviation of the drawdown at the centre of the screen from the average along the screen:

$$\left|\frac{\sigma(1.0;\lambda)-\bar{\sigma}_{u}(\lambda)}{\bar{\sigma}_{w}(\lambda)}\right| \times 100 \qquad (C13)$$

This measure is only valid because the drawdown distribution changes monotonically along the screen away from its centre, for both the uniform and singular flux cases. The results obtained by numerical integration of (C2) and (C8) are shown in Figure C2.

Note that for the singular case the deviation tends to zero as λ, and hence r_w, tends to infinity - as was anticipated. An asymptotic analysis (not presented here) has shown that the deviation tends to zero for the uniform case as λ and (hence) s_w tend to zero. However, another asymptotic result:

$$\lim_{\lambda \to 0} \frac{\sigma(1.0:\lambda)}{\sigma(1.1:\lambda)} = 2 \qquad (C14)$$

indicates that the drawdown at the centre of the screen tends towards twice that at the end of the screen as the diameter tends to zero. This is not inconsistent with the observation that the deviation tends to zero, Figure C2, since the drawdown deviates from the mean value only over an increasingly small region as the diameter tends to zero. (In mathematical terminology, the drawdown distribution tends *uniformly* - but not *pointwise* - to a constant value.)

Note, in particular, from Figure C2, that the drawdown deviation for the singular flux exceeds that for a uniform flux for λ values below about eight. Since, in practice, that would correspond to an unnaturally short screen, the obvious conclusion is that the flux distribution given by (C11) is, in general, of far less validity than the simple uniform distribution. For a confined aquifer the uniformity increases and the conclusion is strengthened

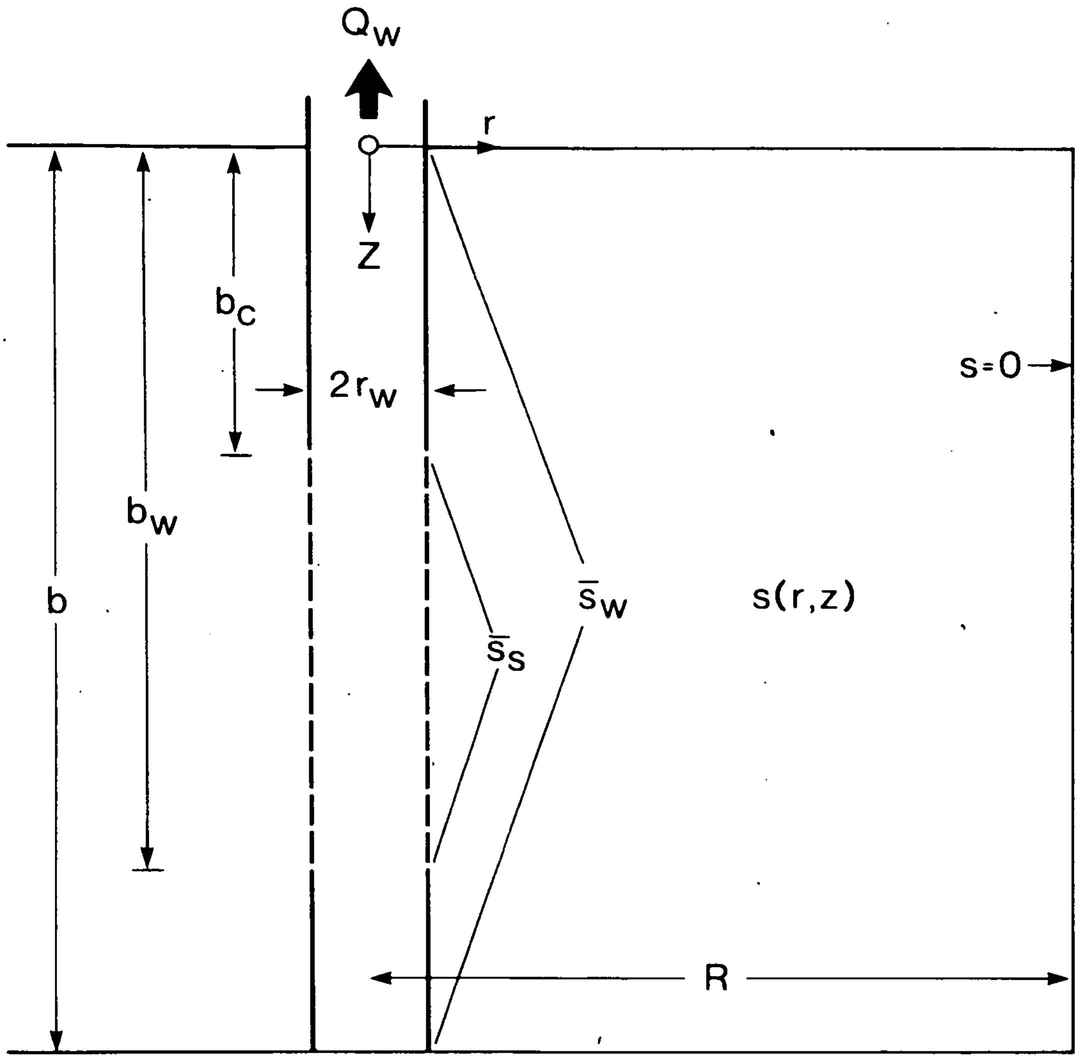

Figure 1. Notation for partial penetration.

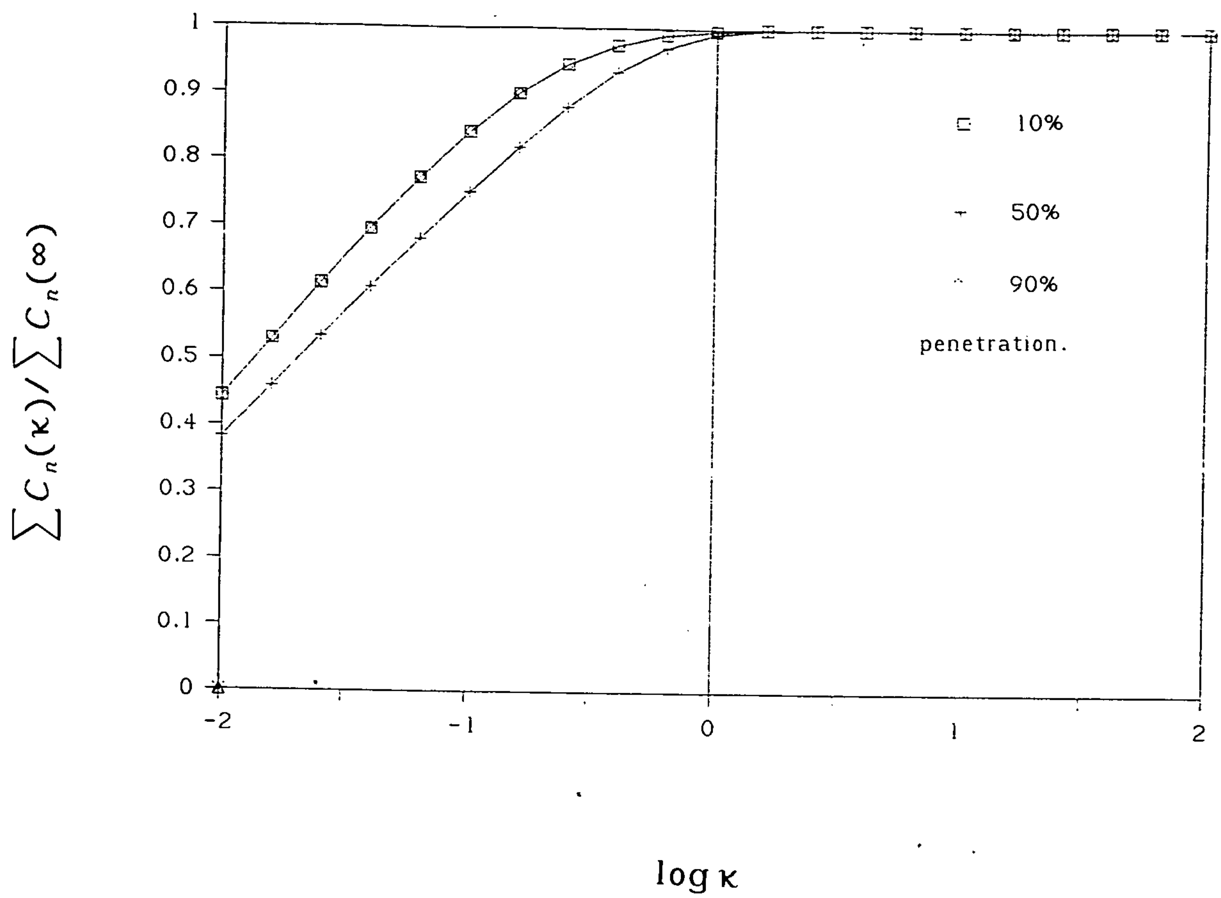

Figure 2. Convergence of equation (5) as the aquifer radius increases, $\kappa \to \infty$. ($\eta = 0$, $\kappa_w = 10^{-3}$)

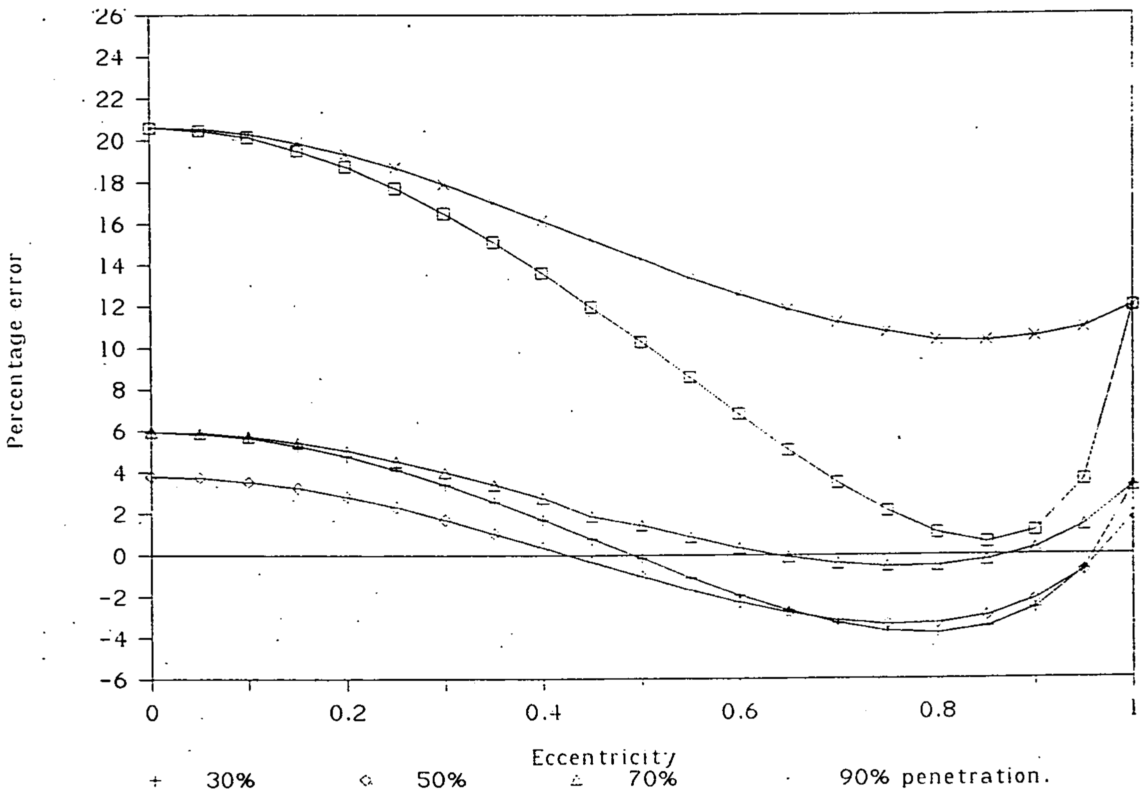

Figure 3a. Error in equation (13) with Θ given by equation (21), $\kappa_w/\pi = 10^{-2}$.

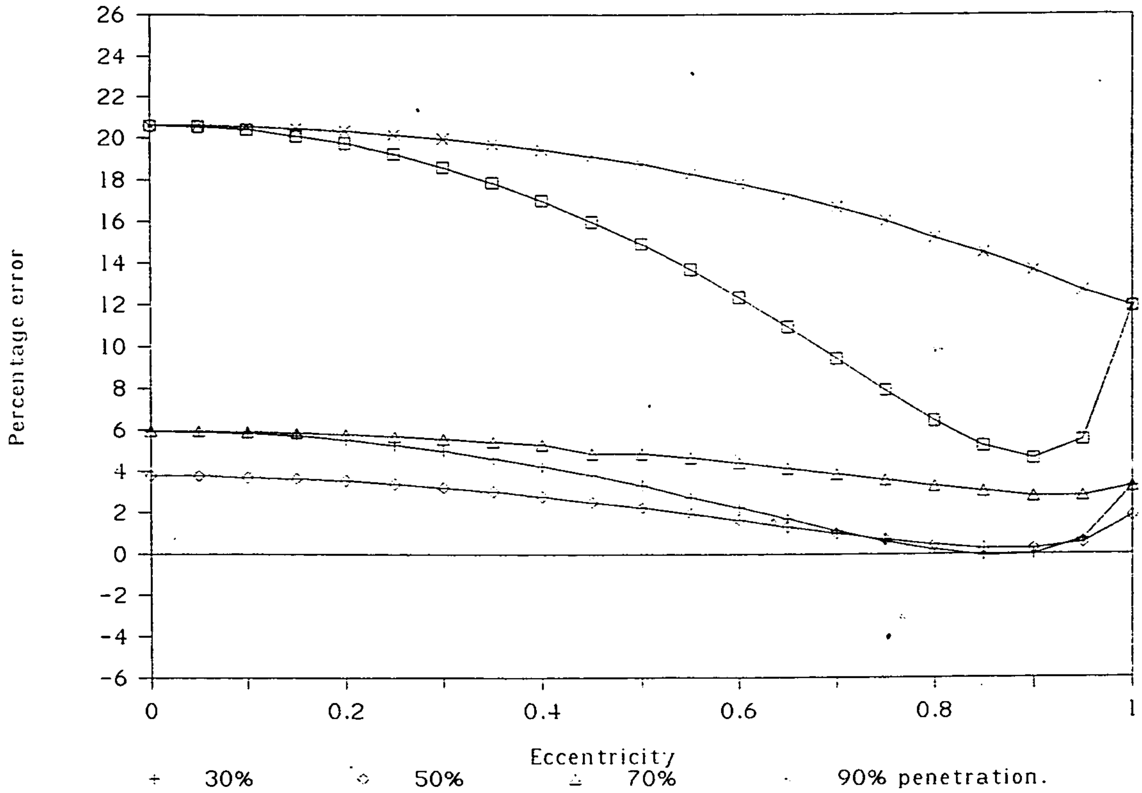

Figure 3b. Error in equation (13) with Θ given by equation (22), $\kappa_w/\pi = 10^{-2}$.

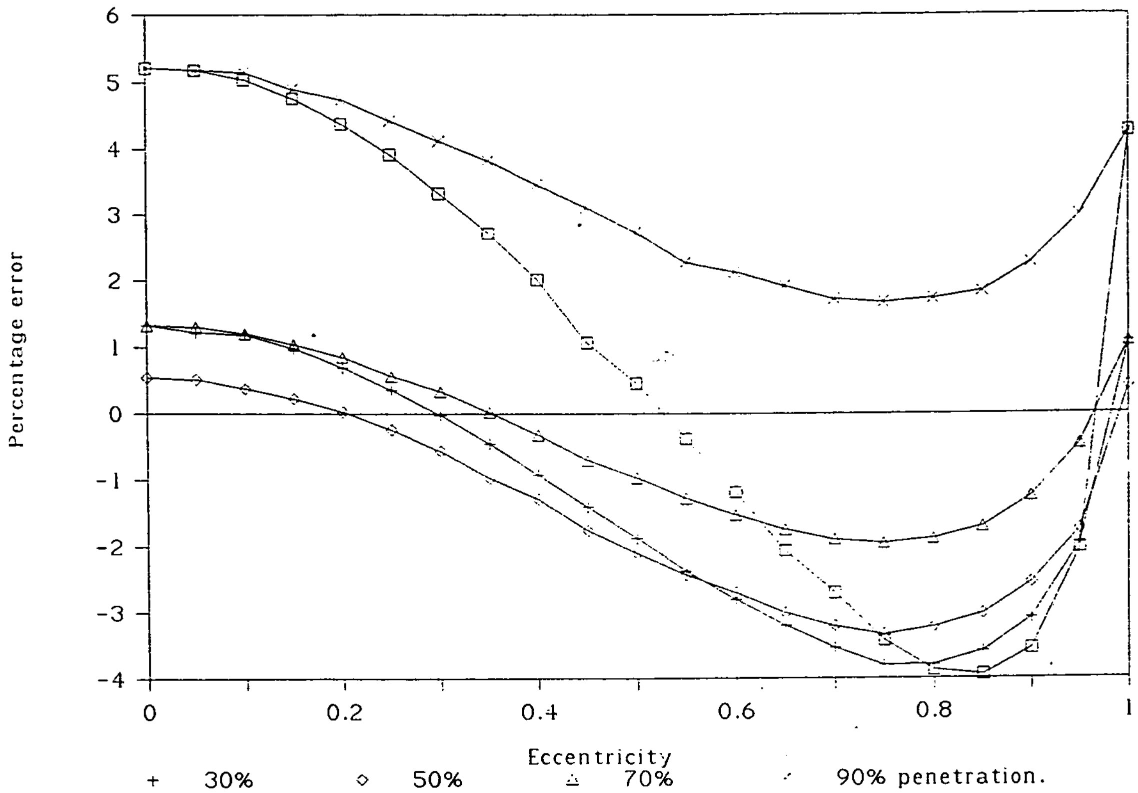

Figure 4a. Error in equation (13) with Θ given by equation (21), $\kappa_w/\pi = 10^{-3}$.

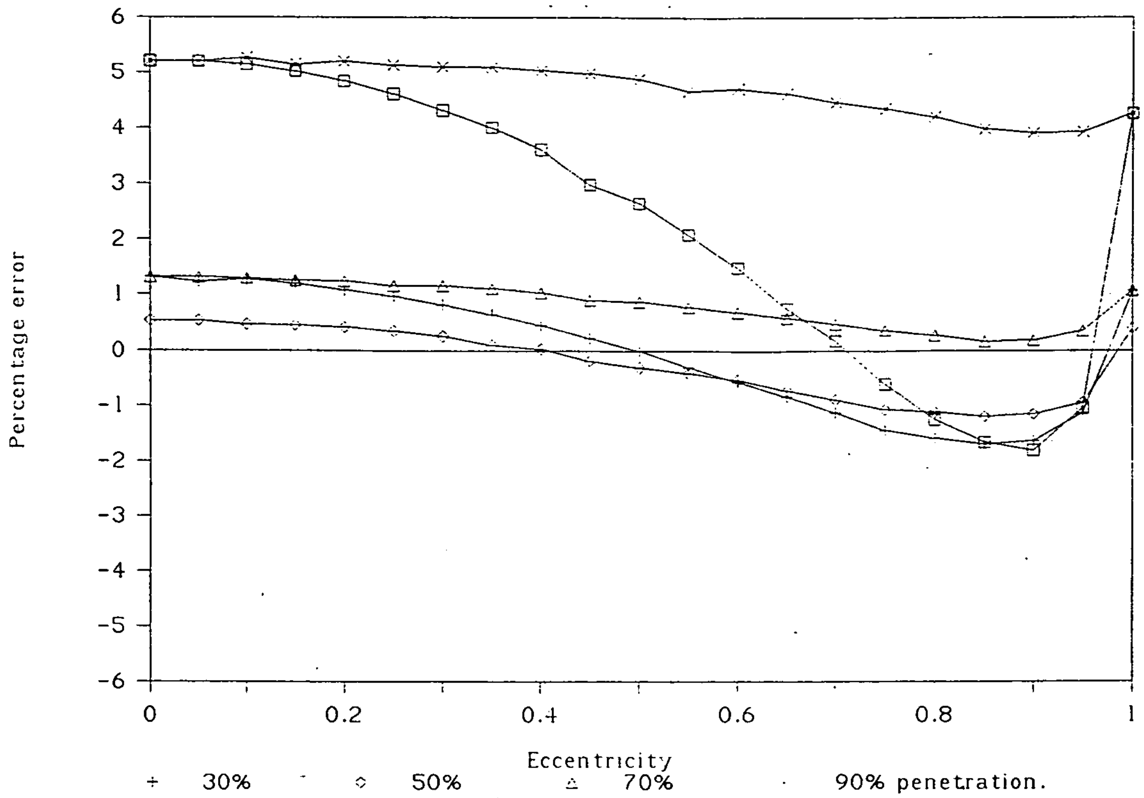

Figure 4b. Error in equation (13) with Θ given by equation (22), $\kappa_w/\pi = 10^{-3}$.

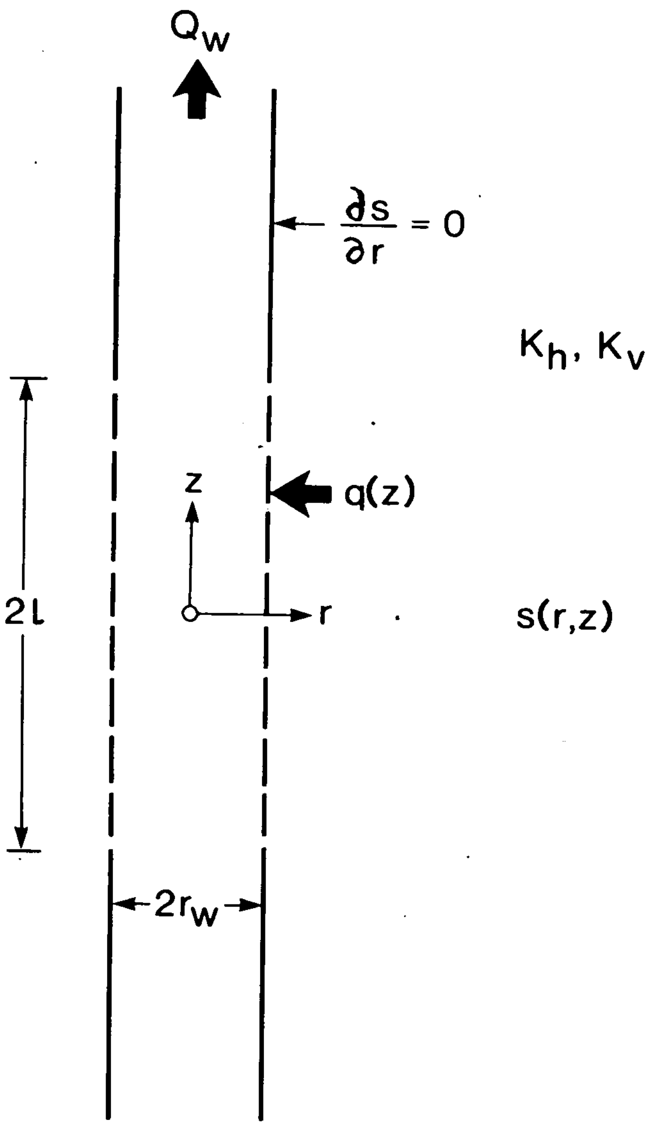

Figure C1. Notation for vertically infinite model for drawdown distribution for a given flow distribution, $q(z)$.

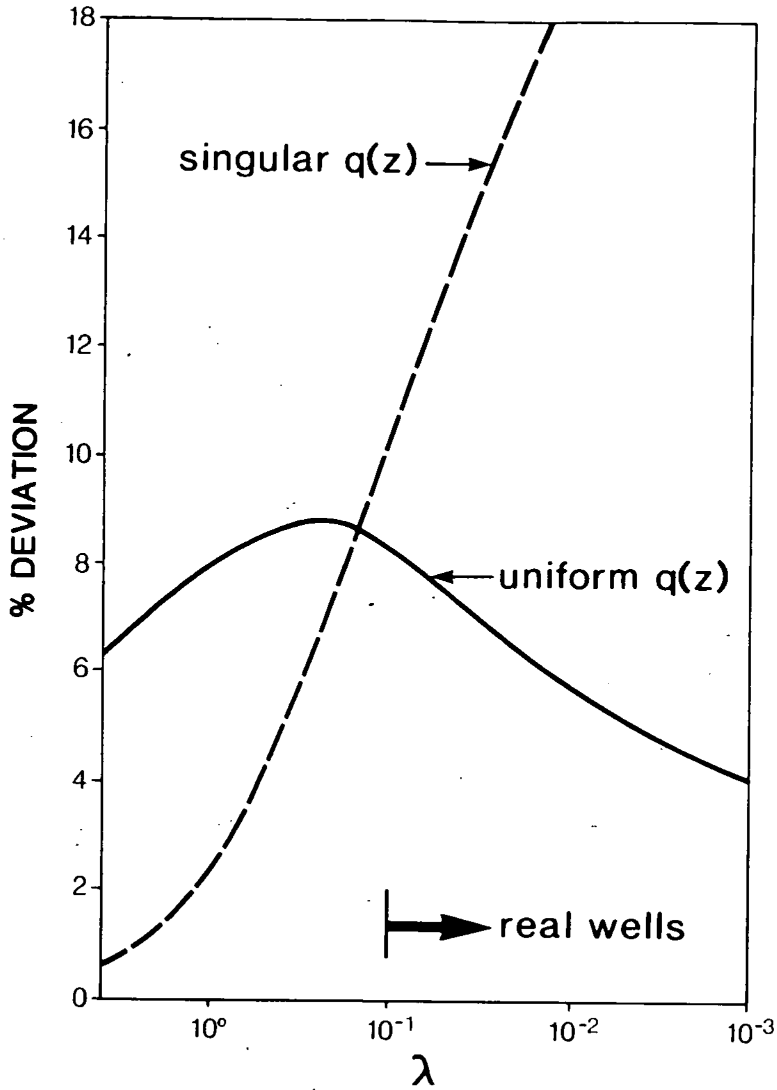

Figure C2. Variation of drawdown along a screen (expressed in terms of equation C13) as a function of the shape factor: $\lambda = \frac{r_w}{l}\sqrt{\frac{K_v}{K_h}}$.