Report to Southern Water Feb 1984

1984/014

MATHEMATICS OF CIRCULAR CHARTS AND THEIR TRANSCRIPTION

INTRODUCTION

The mathematics of circular charts is quite complicated. Their transcription is aggravated further when the relationship between "time" and "angle" is not unique, as is the case when a chart stays on for a whole revolution or more.

Figure 1 provides a definition sketch. With a few minor changes (for Greek letters) the notation of Figure 1 is adhered to in the computer program. Table 1 provides a further check on notation and meanings.

CHART CALIBRATION

The chart centre is not normally marked on the chart and it is therefore necessary to calculate it by reference to three calibration points spaced around the circumference (see Figure 2). Here the circumference of the chart is taken to mean the circle marking the outermost value (usually corresponding to the chart maximum); the radius of this circle is termed the outer radius. The radius of the circle marking the innermost value (usually corresponding to zero) is termed the inner radius. The formula for determining the centre of a circle passing through three given points is derived in Appendix 1. It is assumed that the inner and outer circles are concentric.

The arc line described by the recorder pen usually passes through the chart centre. However, if the pen is misaligned, and for certain types of recorders, this may not be the case; the method used therefore calculates the pen arm radius by a further calibration.

It is convenient to choose a prominent printed arc on the chart - here referred to as the primary arc - for calibration of the pen arm centre and radius (see Figure 2). Three points are chosen: one on the outer circle, one part way down the arc, and one on the inner circle. The formula derived in Appendix 1 is again used to calculate the centre of the collocating circle.

Fig. 1. Definition Sketch for circular chart geometry.

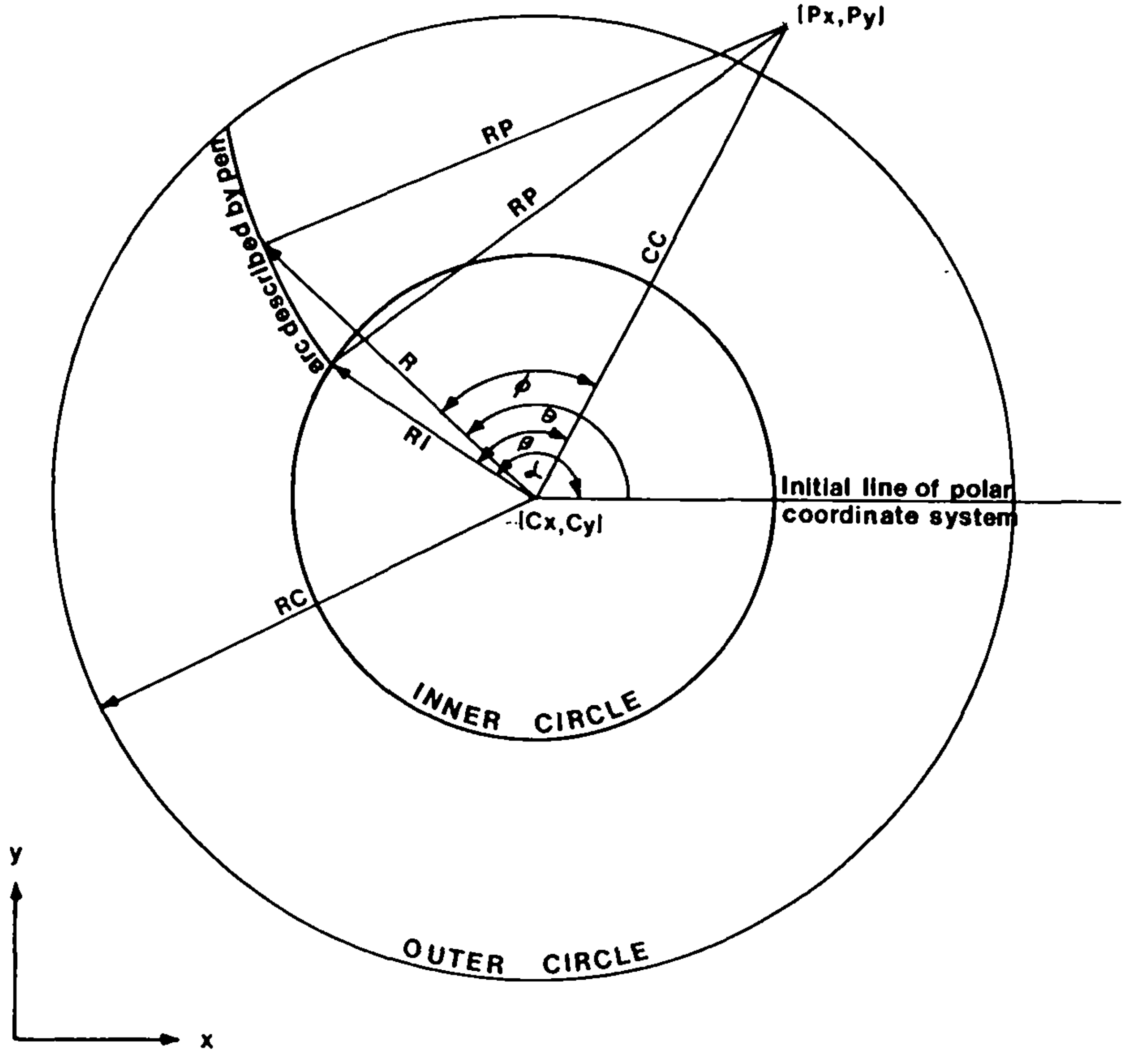
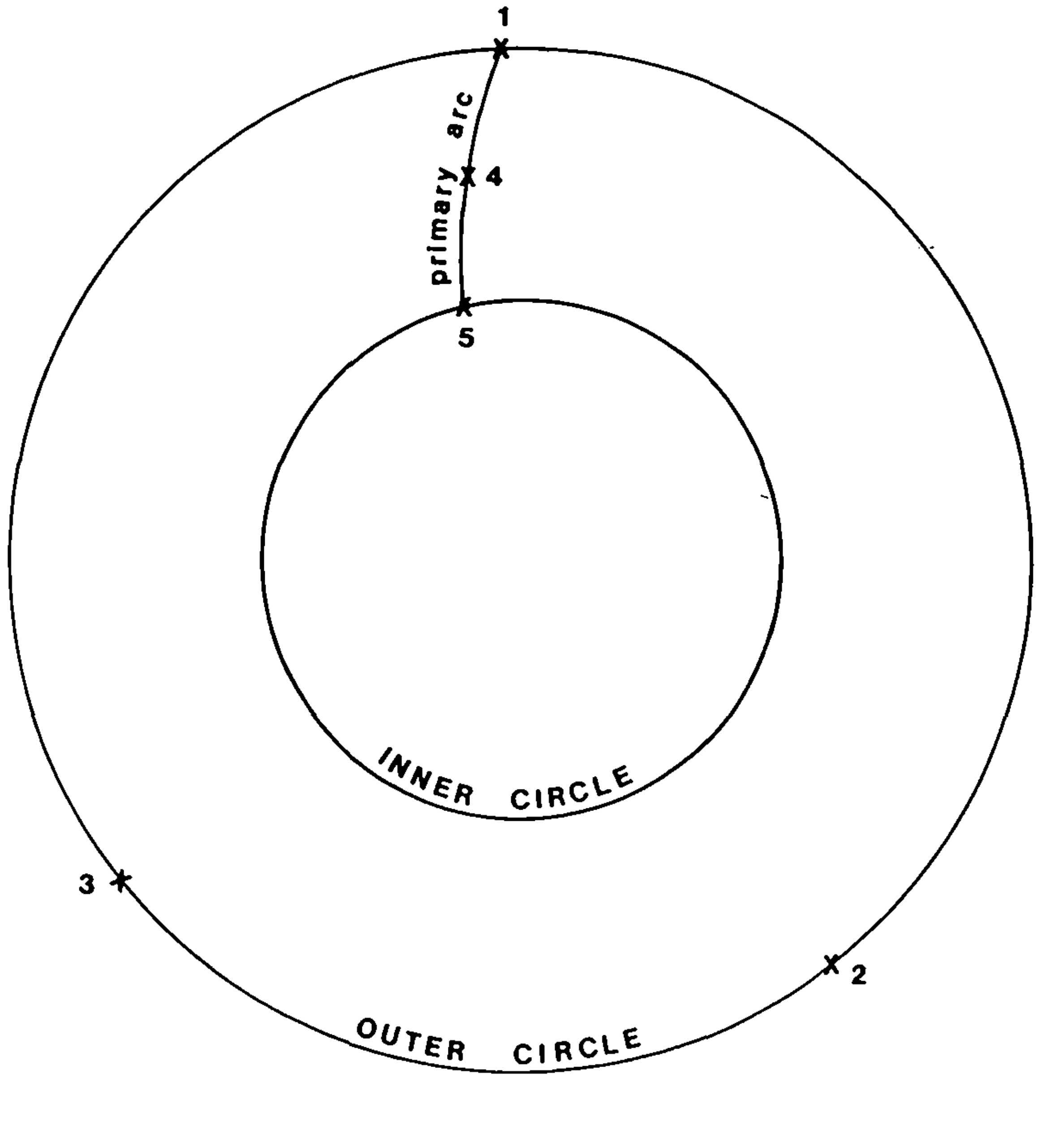


Fig. 2. Definition Sketch for circular chart calibration



X denotes calibration point

It is possible to economize on calibration points by following the arrangement of Figure 2. Calibration point 1 serves two roles, as does calibration point 5. This is because it is necessary to evaluate the inner circle radius RI.

If the chart has several "stages" (ie a radial scale that changes abruptly) or a nonlinear scale, it may be necessary to choose additional calibration points on the primary arc. The program does not include such options but this aspect of chart calibration is no more difficult for circular charts than for rectangular charts. Rather, it is the timing aspect that is tricky.

Chart timing is defined by angular movement of the pen trace about the chart centre. Calibration of the chart timing is determined in the program by inputting the number of hours, HR, corresponding to one revolution of the chart. The conversion factor from angle (in radians) to time (in hours) is:

$$CNV = HR/(2\pi) \qquad (1)$$

HR is generally 168 for a weekly chart but a different value can be input if the operator deduces from chart annotations that the clock mechanism has been running slow or fast. (Reference to the preceding and following charts may assist this interpretation). It is important that the value entered for HR corresponds to one revolution of the chart, not the time for which the particular chart was left "on".

It is convenient at this stage to input a reference time for the first digitized point. If this is set to zero then all times produced by the program will be relative to that of the first digitized point.

CHART TRANSCRIPTION

10. The complexity of circular chart transcription stems from the fact that the angle subtended by the pen arm at the chart centre varies with the radial value marked. In the notation of Figure 1, the angle ϕ varies with the radial distance R.

11. The first step is to convert from cartesian coordinates (x, y) - used by the digitizer - to polar coordinates (R,θ) - relative to the chart centre. As is customary, θ is taken to increase anticlockwise from an "initial line" pointing in the positive x-axis direction (see Figure 1).

12. The conversion to polar coordinates serves to define chart values but not timings. This is because all points of a given radius R have the same value whereas points at a given angle θ do not generally share the same time.

13. The chart timings are determined by projecting each digitized point down an arc line of the chart to a contemporaneous point on the inner circle. The angle then subtended at the polar origin, α, allows the timings to be determined correctly.

14. Referring to Figure 1 it is seen that:

$$\alpha = \theta + \beta \qquad (2)$$

The term $\beta - \phi$ is usually a small positive angle but can be negative for recorders where the pen arm radius is less than the distance between the chart and pen arm centres.

15. Both β and ϕ are calculated by applying the cosine rule to triangles shown in Figure 1:

$$\cos\phi = (R^2 + CC^2 - RP^2)/(2R.CC)$$

and

$$\cos\beta = (RI^2 + CC^2 - RP^2)/(2RI.CC)$$

It should be noted that β is a constant of the chart geometry whereas ϕ varies with R. Both ϕ and β usually lie between about $\pi/3$ and $\pi/2$ radians. Thus it is adequate that the calculation of the inverse cosine, used in the program, yields an angle in the range 0 to π, as is the usual mathematical convention.

16. Equation 2 applies only to charts where the primary arc is convex in the direction of increasing θ (as for Figure 1). This is the case for the Kent type 101 LV/W chart supplied by the customer. However, some charts (eg the British Pitometer Co Ltd type HV934) are concave in the direction of increasing θ. In such situations Equation 2 must be changed to $\alpha = \theta + \phi - \beta$.

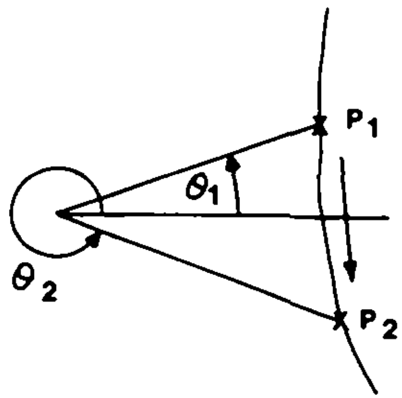
17. The remaining hurdle is to convert the adjusted angle, α, to a time. Various strategies are possible. In what follows we summarize the approach taken in the program and describe the purpose of two specific checks that are made.

18 The first check is to test whether, in passing from one digitized point to the next, the value of θ has jumped from a small angle to a large angle (eg Figure 3a) or from a large angle to a small angle (eg Figure 3b). These situations arise when the "initial line" of the polar coordinate system is crossed. If, as is usual for charts that rotate anticlockwise, it is a Figure 3a case, it is necessary to reduce the angle by 2π radians when calculating the new time. However, because of the curvature of the radial lines on the charts, it is possible for the Figure 3b case to arise occasionally; then, 2π radians must be added when evaluating the new time.

19. The second check is to compare the time corresponding to the new point with that corresponding to the old point. The program prints a warning if a time reversal occurs. In practice it may be preferable to tolerate small reversals

Fig. 3. Illustration of need to adjust for crossings of the "initial line"

a)



θ_1 calculated as $\pi/8$

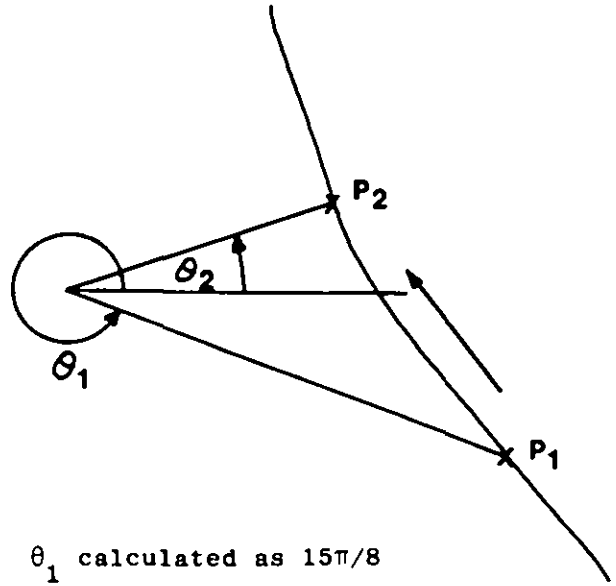
θ_2 calculated as $15\ \pi/8$

Correct angular change is $-\ \pi/4$

which is $15\ \pi/8 - \pi/8 - 2\pi$

or $\theta_2 - \theta_1 - 2\pi$

b)



θ_1 calculated as $15\pi/8$

θ_2 calculated as $\pi/8$

Correct angular change is $+\ \pi/4$

which is $\pi/8 \quad 15\ \pi/8 \quad 2\pi$

or $\theta_2 - \theta_1 + 2\pi$

(say, up to half an hour) but to suppress output of the offending data. The attitude taken may be influenced by the form of interpolation adopted to convert digitized data from an irregular interval to a uniform time interval. This is not a matter specifically related to circular charts.

20. The conversion of angles to times given in the program assumes that the chart revolves anticlockwise (ie the pen marks clockwise). Decreasing polar angles therefore correspond to increasing times. However, it is possible to use the program to transcribe charts that revolve clockwise by simply inputting the number of hours corresponding to one chart revolution as a negative number.

DWR/AC/MAB

TABLE 1. Variable list

CALIBRATION

X,Y	(x,y) coordinates of various calibration points
XX,YY	(x,y) coordinates of various derived points
CX,CY	(x,y) coordinates of chart centre
PX,PY	(x,y) coordinates of pen arm centre
CC	distance between chart and pen arm centres
RC	radius of chart (ie outer radius)
RI	radius of inner circle
RP	radius of pen arm
HR	time in hours corresponding to one revolution of the chart
CNV	conversion factor from angle (in radians) to time (in hours)
BETA	angle subtended by pen arm at chart centre when describing inner circle

TRANSCRIPTION

PHI	φ	angle subtended by pen arm at chart centre
R,THETA	R, θ	polar coordinates of digitized point (relative to chart centre and direction of positive x-axis)
T1	θ_1	value of θ corresponding to previous digitized point
T2	θ_2	value of θ corresponding to current digitized point
		angle obtained by adjusting θ according to Equation 2
AØ	α_o	value of α corresponding to first digitized point
HØ		time in hours corresponding to first digitized point

H1	time in hours corresponding to previous digitized point
H2	time in hours corresponding to current digitized point
CROSS	count of number of crossings of the initial line in the clockwise direction

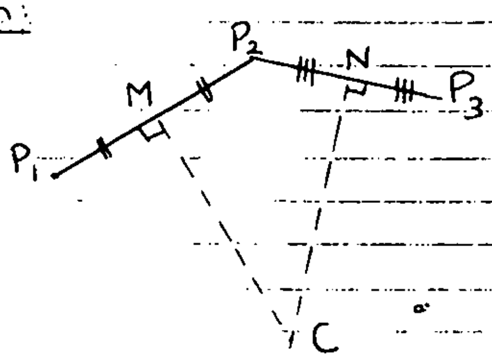
MISCELLANEOUS

PI	
IC	is a function that calculates the inverse cosine in the range 0 to π.
S1	
S2	
MX	See Appendix 1
MY	
NX	
NY	

Appendix 1 — Calculation of centre of circle passing through three given points.

Method: The centre of the circle passing through three points is located by the intersection of the perpendicular bisectors of any two pairs of the points.

Diagram:



$P_1 : (x_1, y_1)$ $M : (x_m, y_m)$

$P_2 : (x_2, y_2)$ $N : (x_n, y_n)$

$P_3 : (x_3, y_3)$ $C : (x_c, y_c)$

Equations $x_m = (x_1 + x_2)/2$; $y_m = (y_1 + y_2)/2$. $x_n = (x_2 + x_3)/2$, $y_n = (y_2 + y_3)/2$

Let gradient of P_1P_2 be S_1. Then $S_1 = (y_2 - y_1)/(x_2 - x_1)$

Hence gradient of MC is $-1/S_1$.

Let gradient of P_2P_3 be S_2. Then $S_2 = (y_3 - y_2)/(x_3 - x_2)$

Hence gradient of NC is $-1/S_2$.

Equation of MC is $y - y_m = -(x - x_m)/S_1$ — (A)

and equation of NC is $y - y_n = -(x - x_n)/S_2$ — (B)

Eliminating y between Equations A and B we obtain:

$$x_c = S_1 S_2 (y_n - y_m + x_n/S_2 - x_m/S_1)/(S_1 - S_2)$$

and, hence, from Equation B

$$y_c = y_n + (x_n - x_c)/S_2$$

as the coordinates of C, the centre of the collocating circle.

ADDITIONAL NOTE

<u>Features that you may wish to incorporate in a complete system for circular chart digitizing</u>

(a) Menus, some options offering to re-use control data entered previously.

(b) Screen prompts to remind the operator how to position the calibration points.

(c) Comparisons with standard chart and recorder properties to provide some quality control of values derived for CC,RC,RI,RP and CNV.

(d) Inclusion of rating relationships; options to deal with multi-stage or nonlinear chart scales.

(e) Real time display of level/flow and time relating to last digitized point.

(f) Inclusion of interpolation routine to provide processed data at a uniform time interval and/or to provide integrated values.

(g) Scaling derived flows to match accumulated reading (on integrator) for the period.

(h) Output of data to file in a standard format.