SOLUTION OF ONE DIMENSIONAL HYDRODYNAMIC EQUATIONS IN TERMS OF AN EXPANSION OF B–SPLINES

by

A .M . DAVIES

REPORT NO. 33

1976

NATURAL ENVIRONMENT RESEARCH COUNCIL
INSTITUTE OF OCEANOGRAPHIC SCIENCES

INSTITUTE OF OCEANOGRAPHIC SCIENCES

Wormley, Godalming,
Surrey, GU8 5UB.
(0428 - 79 - 4141)

(Director: Professor H. Charnock)

Bidston Observatory,
Birkenhead,
Merseyside, L43 7RA.
(051-653-8633)
(Assistant Director· Dr. D. E. Cartwright)

Crossway,
Taunton,
Somerset, TA1 2DW.
(0823-86211)
(Assistant Director: M.J. Tucker)

Marine Scientific Equipment Service
Research Vessel Base,
No. 1 Dock,
Barry,
South Glamorgan, CF6 6UZ.
(04462-77451)
(Officer-in-Charge: Dr. L.M. Skinner)

SOLUTION OF THE ONE DIMENSIONAL HYDRODYNAMIC EQUATIONS IN TERMS OF AN EXPANSION OF B-SPLINES

by

A. M. DAVIES

REPORT NO.33

1976

Institute of Oceanographic Sciences
Bidston Observatory
Birkenhead
Merseyside L43 7RA
England

CONTENTS

		Page
1.	Introduction	1
2.	Model equations and the Galerkin approximation	3
3.	Form of the basis functions and numerical procedure	6
4.	Application of the method to specific problems	11
5.	Extension to problems involving friction, sea bed topography and non-linearities	18
6.	Conclusions	26
	References	27
	Tables I - VII	30

1. INTRODUCTION

The finite difference representation of the hydrodynamic equations of motion has been used extensively in oceanography to calculate approximate solutions to these equations. This numerical method has a number of serious disadvantages in that solutions are only obtained at discrete points in space, and the sea bed topography is represented as a number of point values. Spatial derivatives are usually calculated in terms of simple differences of two grid point values, and although higher order differencing methods have been used [14] involving up to four or six grid point values, this accuracy can only be applied to systems having periodic boundary conditions, and in problems having boundary conditions where either elevation or current must be specified, a one sided lower order representation has to be applied at the boundary.

Generally the grid involved in finite difference work is uniform over the region for which a solution is to be determined, and methods which aim at improving resolution in a particular interior region, by inserting a fine mesh within the coarser one have been shown mathematically [15] to have a number of undesirable effects. Namely, changes in the phase and amplitude of a wave progressing from one mesh to another; aliasing problems involving transfer of energy from waves representable on the fine mesh (but not on the coarse mesh) to longer wavelengths. Reflection of incident waves, and generation of spurious waves at the boundary between the two meshes and also numerical instability of the coupled system, are particularly difficult problems.

In order to overcome some of these problems this paper presents a method involving the solution of the hydrodynamic equations of motion using an expansion in terms of B-splines. These functions are piecewise polynomials, being non zero only over a small region of the space domain. This method yields continuous solutions in space for both sea surface elevation and current, and since the solution is calculated in terms of an expansion, accurate values of the sea surface slope at any point can be readily obtained by direct differentiation. The sea bed topography is also represented as a continuous spatial function.

The piecewise nature of these functions permits boundary conditions to be readily incorporated without degrading the accuracy of the solution near the boundary. In regions where the sea bed topography is particularly complex, or greater accuracy of the solution is required, more functions can be incorporated to sharpen resolution, and since the method represents spatial derivatives analytically in terms of the complete expansion, and not as differences of grid point values, the problem of reflection or generation of spurious waves at the boundaries of regions of different resolution is avoided.

The Galerkin [1] method, in which the error between the true solution, and the solution in terms of an expansion of basis functions (in this case B-splines) is minimized in a least squares sense, is used to solve the hydrodynamic equations.

This technique has been used extensively in the form of the finite element method, in solving the partial differential equations arising in structural mechanics, but it is only lately that it has been applied to solving time dependent problems arising in meteorology and oceanography.

Orszag [2,3] has used the spectral method, in which a solution is obtained in terms of functions which are continuous over the whole spatial domain (usually Chebyshev polynomials, Legendre functions or a Fourier expansion) to solve a number of meteorological problems.

The finite element method, in which the basis consists of low order piecewise polynomials, has been applied by Cullen [4] to the integration of the shallow water equations on a sphere. The advantages of the finite element method, in terms of computer storage, time and accuracy have been demonstrated by Wang et al [5].

In oceanography Rao [6] proposes a type of spectral method for the calculation of storm surges in a lake, using the normal modes of the lake as his basis functions. The finite element method has been applied by Grotkop [7] to the calculation of the M_2 tidal constituent in the North Sea.

2. MODEL EQUATIONS AND THE GALERKIN APPROXIMATION

In one dimension the equations of motion and continuity for a single homogeneous layer of fluid, neglecting friction and advective terms, are given by :-

$$\frac{\partial U}{\partial t}(x,t) + g\frac{\partial \xi}{\partial x}(x,t) = 0 \tag{1}$$

and

$$\frac{\partial \xi}{\partial t}(x,t) + H\frac{\partial U}{\partial x}(x,t) = 0 \tag{2}$$

where H , a constant, is the mean depth of fluid, U is the fluid speed in the x direction, ξ the total depth of the fluid, and g the acceleration of the Earth's gravity. In order to obtain unique solutions for (1) and (2), initial conditions $U(x,0)$ and $\xi(x,0)$ at time $t=0$ and boundary conditions must be specified. The form of these is considered later, when the method is applied to a number of problems.

Instead of approximating these equations directly, as in the case of finite differences (where the derivatives are replaced by differences, giving a system of algebraic equations), the

Galerkin approximation aims at minimizing the functional obtained by multiplying these equations by test function $V(x)$ belonging to a space S of continuous differentiable functions, and integrating over the region for which a solution is required.

Thus multiplying (1) and (2) by $V(x)$ and integrating from 0 to L, where $x=0$ to $x=L$ defines the region over which solutions for U and ξ are required, gives

$$\int_0^L \left(\frac{\partial U}{\partial t}(x,t) + g\,\frac{\partial \xi}{\partial x}(x,t)\right) V(x)\,dx = 0 \tag{3}$$

and
$$\int_0^L \left(\frac{\partial \xi}{\partial t}(x,t) + H\,\frac{\partial u}{\partial x}(x,t)\right) V(x)\,dx = 0 \tag{4}$$

In practice a subspace S_p of S is used, U and ξ, being expanded in terms of a basis $V_1(x), V_2(x), \ldots\ldots V_p(x)$ and time dependent coefficients as :-

$$U(x,t) = \sum_{i=1}^{p} \alpha_i(t)\,V_i(x) \tag{5}$$

$$\xi(x,t) = \sum_{i=1}^{p} \beta_i(t)\,V_i(x) \tag{6}$$

These solutions must satisfy the initial and boundary conditions imposed in solving (1) and (2).

Substituting (5) and (6) into (3) and (4), taking $V(x) = V_j(x)$, $j = 1, 2, \ldots, P$ gives :

$$\sum_{i=1}^{p} \frac{d\alpha_i}{dt} \int_0^L V_i V_j \, dx + g \sum_{i=1}^{p} \beta_i \int_0^L \frac{dV_i}{dx} V_j \, dx = 0 \tag{7}$$

$$\sum_{i=1}^{p} \frac{d\beta_i}{dt} \int_0^L V_i V_j \, dx + H \sum_{i=1}^{p} \alpha_i \int_0^L \frac{dV_i}{dx} V_j \, dx = 0 \tag{8}$$

where j = 1,2,...p

Solving this set of coupled differential equations for the α's and β's through time gives an approximate solution to equations (1) and (2). In order to solve this set of equations, initial values, $\alpha_i(0)$ and $\beta_i(0)$ have to be calculated.

In the case of $U(x,0)$ and $\xi(x,0)$ having functional forms $U(x)$ and $Z(x)$, using (5) and (6) the latter can be expanded in terms of the p basis functions and then employed to give :

$$\sum_{i=1}^{p} \alpha_i(0) \int_0^L V_i V_j \, dx = \int_0^L U(x) V_j \, dx \tag{9}$$

and

$$\sum_{i=1}^{p} \beta_i(0) \int_0^L V_i V_j \, dx = \int_0^L Z(x) V_j \, dx \tag{10}$$

$j = 1, 2, \dots p$

For simple functions $U(x)$ and $Z(x)$, the integrals occurring in (9) and (10) can be evaluated analytically, and the system of equations solved for the $\alpha_i(0)$ and $\beta_i(0)$. For well behaved $U(x)$ and $Z(x)$ the integrals can be evaluated using Gauss quadrature. When $U(x)$ and $Z(x)$ are only given at m distinct points $(m \geq p)$, then the least squares criterion can be adopted in fitting the expansions (5) and (6) to the points, yielding the usual set of normal equations from which the $\alpha_i(0)$'s and $\beta_i(0)$'s can be obtained.

3. FORM OF THE BASIS FUNCTIONS AND NUMERICAL PROCEDURE

In order to solve the set of differential equations given in (7) and (8), it is necessary to evaluate all the required integrals over the basis functions. The ease with which values of these are obtained depends upon the nature of the basis set. Spline functions can be readily integrated and differentiated, and because of their piecewise nature, many of the integrals that occur, are zero.

The basis functions used here are the B-splines (fundamental splines), which were first studied by Schoenberg [8] and have recently been used extensively in problems of interpolating and smoothing by Powell [9] and Schumaker [10] .

The most common representation of a spline $S(x)$ of order n , (degree $n-1$) based on an increasing set of knots $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots \lambda_N$ is that given by Ahlberg, Nilson and Walsh [11] :

$$S(x) = \sum_{i=1}^{n} d_i x^{i-1} + \sum_{j=1}^{N} e_j (x-\lambda_j)_+^{n-1} \tag{11}$$

where

$$(x-\lambda_j)_+^{n-1} = \begin{cases} 0 & \text{when } (x-\lambda_j) < 0 \\ (x-\lambda_j)^{n-1} & (x-\lambda_j) \geq 0 \end{cases} \tag{12}$$

This particular form of the spline, however, gives rise to badly conditioned matrices when used in numerical work, and an expansion in terms of B-splines is much more desirable. In this case it is necessary to introduce $2n$ additional knots $\lambda_{1-n}, \lambda_{2-n}, \ldots\ldots \lambda_0$ and $\lambda_{N+1}, \lambda_{N+2}, \ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots \lambda_{N+n}$, then for $\lambda_0 \leqslant x \leqslant \lambda_{N+1}$, $S(x)$ can be expressed as

$$S(x) = \sum_{i=1}^{N+n} c_i M_{ni}(x) \tag{13}$$

where $M_{ni}(x)$ is the B-spline of order n defined upon knots $\lambda_{i-n}, \lambda_{i-n+1}, \ldots\ldots\ldots \lambda_i$ with the restricted support property

$$M_{ni}(x) \begin{cases} > 0 & \text{when } \lambda_{i-n} < x < \lambda_i \\ = 0 & x < \lambda_{i-n}, x > \lambda_i \end{cases} \tag{14}$$

Thus $M_{ni}(x)$ is only non zero over n knot intervals.

For the region of interest $0 \leqslant x \leqslant L$, it is necessary to have $2n$ knots outside the region to provide support for the spline functions, positioned at,

$$\begin{aligned} &\lambda_{1-n} \leqslant \lambda_{2-n} \ldots\ldots \leqslant \lambda_0 = 0 \\ \text{and} \quad &L = \lambda_{N+1} \leqslant \lambda_{N+2} \ldots\ldots \leqslant \lambda_{N+n} \end{aligned} \tag{15}$$

where N is the number of knots within the region.

These spline functions can readily be evaluated by a numerically stable method given by Cox [12], using the recurrence relationship :-

$$M_{\tau i}(x) = \frac{(x-\lambda_{i-\tau})M_{\tau-1,i-1}(x) + (\lambda_i - x)M_{\tau-1,i}(x)}{\lambda_i - \lambda_{i-\tau}} \tag{16}$$

for $\tau = 2, 3, \ldots\ldots n$

starting with

$$M_{1i}(x) = \begin{cases} 1/(\lambda_i - \lambda_{i-1}) & \text{when} \quad \lambda_{i-1} \leq x < \lambda_i \\ 0 & \qquad\quad x < \lambda_{i-1},\ \lambda_i \leq x \end{cases} \tag{17}$$

The detailed computational procedure given by Cox [12] was used for this.

By using this formulation the knots of the spline may be specified in arbitrary positions, and may coincide with one another. If k knots coincide, then a spline $S(x)$ has in general only $n-k-1$ continuous derivatives (n being the order), and for $n-1$ coincident knots only $S(x)$ is continuous.

Thus the B-splines form a very convenient basis in which the current $U(x,t)$ and elevation $\xi(x,t)$ can be expanded. In order to solve the system of equations given in (7) and (8), it is necessary to evaluate integrals of the form

$$\int_0^L M_{ni}(x)\, M_{nj}(x)\, dx \quad \text{and} \quad \int_0^L \frac{d}{dx} M_{ni}(x)\, M_{nj}(x)\, dx \tag{18}$$

Rather than calculate these integrals directly, it is convenient to express the splines in terms of Chebyshev polynomials. Over each knot interval $\lambda_j \leq x \leq \lambda_{j+1}$ the B-spline is a polynomial of degree $n-1$ or less, and can be expressed as

$$M_{nj}(x) = \sum_{i=0}^{n-1}{}'' \gamma_{ji} T_i(X) \tag{19}$$

where $X = (2x - \lambda_j - \lambda_{j+1})/(\lambda_{j+1} - \lambda_j)$, for $-1 \leq X \leq 1$, and $T_i(X)$ is a Chebyshev polynomial of the first kind. The double prime indicates that the first and last γ's are to be halved when the sum is evaluated. The γ_{ji} is given by (Fox and Parker [13])

$$\gamma_{ji} = \frac{2}{n-1} \sum_{k=0}^{n-1}{}'' T_i(X_k) M_j(x_k) \quad (i = 0, 1, \dots n-1) \tag{20}$$

with $X_k = \cos \dfrac{k\pi}{n-1}$ and $x_k = \dfrac{X_k(\lambda_{j+1} - \lambda_j) + \lambda_j + \lambda_{j+1}}{2}$

Due to the piecewise nature of the B-splines, many of the integrals which occur in (18) will be zero. Expressing these as sums of integrals evaluated between knots, gives rise to further zero integrals. Using the transformation given in (19), the integrals can be expressed in terms of integrals of Chebyshev polynomials.

$$\frac{\lambda_{j+1} - \lambda_j}{2} \int_{-1}^{+1} T_r(X)\, T_s(X)\, dX \quad \text{and} \quad \int_{-1}^{+1} T_r(X) \frac{dT_s}{dX}(X)\, dX \tag{21}$$

Making the substitution $X = \cos\theta$ and using :-

$$T_r(X)\,T_s(X) = 0{\cdot}5\left(T_{s+r}(X) + T_{|s-r|}(X)\right) \tag{22}$$

$$T_r(X) = \cos(r \cos^{-1} X) \tag{23}$$

these integrals can be readily evaluated.

For high values of r and s, which occur for large n, the Fast Fourier transform method, can be used to compute the integrals in a similar manner to that used in spectral methods [2,3].

The set of coupled differential equations (7) and (8), using a basis of B-splines can be written in matrix form as

$$\begin{aligned} \mathbb{C}\frac{d\boldsymbol{\alpha}}{dt} + g\,\mathbb{A}\,\boldsymbol{\beta} &= \mathbb{O} \\ \mathbb{C}\frac{d\boldsymbol{\beta}}{dt} + H\,\mathbb{A}\,\boldsymbol{\alpha} &= \mathbb{O} \end{aligned} \tag{24}$$

where $\mathbb{C}$ and $\mathbb{A}$ are $p \times p$ matrices, with (i,j) elements given by

$$C_{ij} = \int_0^L M_i\,M_j\,dx \quad \text{and} \quad A_{ij} = \int_0^L \frac{dM_i}{dx}\,M_j\,dx \tag{25}$$

where M_i and M_j are B-splines of order n, the subscript n having been dropped from the notation. The matrix $\mathbb{C}$ is readily inverted using Gauss elimination, and any standard numerical method can be used to integrate the differential equations. However in order to keep the errors arising from solving these equations at discrete time steps a minimum, a fourth order Runge Kutta technique was used.

4. APPLICATION OF THE METHOD TO SPECIFIC PROBLEMS

(a) Periodic boundary conditions

The numerical example used by Wang et al [5] provides a good test of the application of the method to a problem having periodic boundary conditions.

The initial conditions are :-

$$u(x,0) = U_0 \sin 2\pi r x/L$$

and $$\xi(x,0) = H \qquad (26)$$

Boundary conditions are given by :-

$$\begin{aligned} u(x,t) &= u(x+L,t) \\ \xi(x,t) &= \xi(x+L,t) \end{aligned} \qquad (27)$$

where U_0 is a constant amplitude, r is an integer determining the wavelength, H , a constant, is the depth of fluid, taken as 9.184 km, and L = 10,500 km, g = 9.81 m/sec^2.

Using cubic B-splines (n = 4) as a basis, it is necessary to provide additional knots outside the region $0 \leq x \leq L$ to support these splines, given by :-

$$\lambda_{-3} < \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \ldots\ldots\ldots L = \lambda_{N+1} < \lambda_{N+2} < \lambda_{N+3} < \lambda_{N+4}$$

where $$\begin{aligned} \lambda_{-1} &= \lambda_0 - \Delta\lambda_{N+1}, & \lambda_{N+2} &= \lambda_{N+1} + \Delta\lambda_1 \\ \lambda_{-2} &= \lambda_{-1} - \Delta\lambda_N, & \lambda_{N+3} &= \lambda_{N+2} + \Delta\lambda_2 \\ \lambda_{-3} &= \lambda_{-2} - \Delta\lambda_{N-1}, & \lambda_{N+4} &= \lambda_{N+3} + \Delta\lambda_3 \end{aligned} \qquad (28)$$

with $\Delta\lambda_i = \lambda_i - \lambda_{i-1}$

The relationships given by (28) for the knot positions, take account of the periodic nature of the boundary conditions, if integrals of the form C_{ij} and A_{ij} given by equation (25) are replaced by,

$$\int_0^L M_i M_j \, dx + \int_0^L M_{i+N+1} M_{j+N+1} \, dx \quad \text{for } i,j = 1,2,3 \tag{29}$$

and

$$\int_0^L \frac{d}{dx} M_i M_j \, dx + \int_0^L \frac{d}{dx} M_{i+N+1} M_{j+N+1} \, dx \quad \text{for } i,j = 1,2,3$$

To test the accuracy of the method $\xi(x,t)$ and $U(x,t)$ were calculated at 200 equally spaced points in the region $0 \leqslant x \leqslant L$, at each time step, and the maximum differences between these and the analytic solution given by Wang et al [5], were obtained. The maximum value of the error in $\xi\,(\Delta\xi)$ and $U(\Delta U)$ (normalized by dividing by H and U_0 respectively) which occurred during each integration period is given in the tables.

Table I presents the results for Υ = 1.0, U_0 = 54.6 m/sec, using a cubic B-spline (n = 4) in both single precision (s.p., 7-figure accuracy) and double precision (d.p., 16-figure accuracy). A time step of 600 seconds (approximately one sixtieth of the period) was used, except in the case of 18 basis functions where a smaller time step (300 secs) was required to prevent numerical errors in the time integration having a large effect. These results illustrate the increase in accuracy that can be obtained as the number of basis functions is increased. As the size of the basis set grows, precision problems associated with inversion of the matrix $\mathbb{C}$, and solution of the system of equations become more important, as indicated by the deterioration in the single precision results.

In Table II results for the same problem are presented using a sixth order B-spline ($n = 6$). Comparing the double precision results in Tables I and II it is evident that increasing the spline's order improves the accuracy of the result.

Table III shows the normalized results obtained when $\tau = 2.0$, $U_0 = 27.3$ m/sec, in this case there are two wavelengths present in the region $0 \leq x \leq L$, and the number of basis functions must be increased in order to represent the wave correctly. Increasing the order of the spline again improves the accuracy of the results.

Although these calculations were performed for equally spaced knots, the method for evaluating the B-splines allows for unequal knot intervals.

(b) Specification of elevation and current at the same boundary.

Periodic boundary conditions as defined in the preceding problem are rarely encountered in oceanography.

An example of more realistic character is concerned with a progressive wave in a channel, where the elevation and current on the boundary $x = 0$ vary in a periodic manner, given by

$$\begin{aligned} \xi(0,t) &= C \cos \omega t = Z_0(t) \\ u(0,t) &= AC \cos \omega t = U_0(t) \end{aligned} \tag{30}$$

The analytic solution of (1) and (2) subject to these boundary conditions are given by

$$\begin{aligned} \xi(x,t) &= C \cos(\tau x - \omega t) \\ u(x,t) &= AC \cos(\tau x - \omega t) \end{aligned} \tag{31}$$

where $A = (g/H)^{1/2}$, τ determines the wavelength, and $\omega = \tau (gH)^{1/2}$ the period. The solution represents a progressive wave travelling along the channel from $x = 0$ to $x = L$.

The implementation of boundary conditions can be accomplished by two methods. If the knot at $x = 0$ has a multiplicity of $n-1$, then all the B-splines in expansions (5) and (6) except the first will vanish at this point. Thus the expansions for ξ and U which satisfy the boundary conditions become

$$U(x,t) = \frac{U_0(t)}{A_1} M_1(x) + \sum_{i=2}^{p} \alpha_i M_i(x) \tag{32}$$

and

$$\xi(x,t) = \frac{Z_0(t)}{A_1} M_1(x) + \sum_{i=2}^{p} \beta_i M_i(x) \tag{33}$$

where $A_1 = M_1(0)$, and p is the number of basis functions.

The unknown coefficients α_i and β_i being determined as previously, using $M_2, M_3, \ldots\ldots M_p$ as test functions giving $(p-1) \times (p-1)$ matrices $\mathbb{A}$ and $\mathbb{C}$.

In the second method, a single knot is placed at $x = 0$. In this case the first $n-1$ B-splines will be non-zero at this point, and in order to satisfy the imposed boundary conditions the differential equations given by (7) and (8) must be solved subject to the constraints imposed by the boundary conditions on the first $n-1$ α's and $n-1$ β's namely :

$$\alpha_1 A_1 + \alpha_2 A_2 + \ldots \alpha_{n-1} A_{n-1} = U_0(t) \tag{34}$$

and

$$\beta_1 A_1 + \beta_2 A_2 + \ldots \beta_{n-1} A_{n-1} = Z_0(t) \tag{35}$$

where $A_i = M_i(0)$

In the previous problem, where periodic boundary conditions were used, these conditions imposed the constraint that the wave had the same amplitude and phase at $x=0$ and $x=L$. In this problem that constraint is no longer required, and values of τL which are not integer multiples of π were chosen to ensure different amplitudes and phases at $x=0$ and $x=L$, namely (i) $\tau L = 1{\cdot}6\pi$, (ii) $3{\cdot}2\pi$, (iii) $6{\cdot}4\pi$. The value of τL is important since from (31) it determines the number of wavelengths present and hence the number of basis functions required to describe the wave accurately.

Initial conditions corresponding to (31) with $t=0$ were used, together with a time step of one sixtieth of the period. After a few cycles, the error reached a value determined by the periodic forcing of the boundary conditions, and the rounding error in the calculations. Table IV gives the normalized maximum error found at 200 points in the elevation for case (ii) $\tau L = 3{\cdot}2\pi$. The current is related to the elevation by the constant A, and shows that the error is fairly constant after the first few cycles.

Identical calculations were performed for the three waves, cases (i), (ii) and (iii), with both fourth and sixth order B-splines, the results after 15 cycles are summarized in Table V, illustrating that as the number of wavelengths in the region $0 \leq x \leq L$ increases the number of basis functions must also increase in order to represent the wave accurately, although increasing the order of the spline can have a similar effect.

Calculations for all cases were also repeated using the method of constraining the expansion coefficients to satisfy the boundary conditions, and gave results which agreed with Table V.

(c) Elevation and current specified at opposite boundaries

Consider a gulf connected to a tidal ocean, with the closed end of the gulf at $x = 0$ and the open end at $x = L$. At $x = L$ there is a forced sinusoidal variation in the elevation. The appropriate boundary conditions are

$$
\begin{aligned}
U(0,t) &= 0 \quad \text{for all } t \\
\xi(L,t) &= A \cos \omega t
\end{aligned}
\tag{36}
$$

Analytic solutions of (1) and (2) with these boundary conditions give

$$
\begin{aligned}
U(x,t) &= AC \frac{\sin \tau x}{H \cos \tau L} \sin \omega t \\
\xi(x,t) &= A \frac{\cos \tau x}{\cos \tau L} \cos \omega t
\end{aligned}
\tag{37}
$$

where A is the amplitude of the forcing sinusoid, ω the frequency, $C = (gH)^{1/2}$, and $\tau = \omega / C$ determines the wavelength. Only the value of τL is important in determining the number of basis function required, however since the problem is one that occurs frequently in oceanography, typical values of the other parameters were used, namely L = 300 km, H = 90.8m, g = 9.81m/sec^2, A = 1.0m.

Two calculations were performed using (i) τL = 6.0 and (ii) τL = 12.0. Initial conditions corresponding to (37) with $t = 0$ were used, together with a time step one sixtieth of a period. After a few cycles the error reached a nearly constant value, and the maximum errors $\Delta \xi$ and ΔU after 15 cycles are given in Table VI.

The boundary conditions were satisfied by applying constraints to the expansion coefficients as described previously. This example is of interest not only because boundary conditions are applied to ξ and U at opposite ends of the channel, but unlike the case of the progressive wave, the elevation and current are out of phase in both space and time. The results show that as in previous examples, accurate solutions can be obtained by increasing the number of basis functions and order of the spline.

(d) <u>Current specified on opposite boundaries</u>

Analytic solutions of equations (1) and (2) for oscillations of a body of water in a channel closed at both ends ($U(0,t) = U(L,t) = 0$ for all t) are given by

$$U(x,t) = C \sin \frac{r\pi x}{L} \sin \frac{2\pi t}{T} \tag{38}$$

$$\xi(x,t) = \frac{TH}{2L} C \cos \frac{r\pi x}{L} \cos \frac{2\pi t}{T}$$

where C is the amplitude and $T = 2L/(gH)^{1/2}$ is the period. Numerical values used were L = 300 km, H = 90.8m, g = 9.81m/sec^2 and C = 1.0m/sec the initial conditions are given by (38) with t = 0, a time step one sixtieth of the period being employed in the integration. Two calculations were performed with (i) r = 1.0, (ii) r = 2.0, and maximum errors after 15 cycles are presented in Table VII. To check the stability of the results the calculation was continued for over 100 cycles, and results consistent with Table VII were obtained. The boundary conditions of zero flow were readily incorporated by using a knot of multiplicity $n-1$ at the points $x=0$ and $x=L$.

The examples given in this section illustrate the ease with which various boundary conditions involving specification of elevations, currents, and periodic conditions can be incorporated.

5. EXTENSION TO PROBLEMS INVOLVING FRICTION, SEA-BED TOPOGRAPHY, AND NON-LINEARITIES.

Although the results presented previously deal with a physically idealized situation, i.e. a channel of constant depth with no friction, they are particularly important in demonstrating the accuracy of the method, and its numerical stability. The technique is here extended to the equations of motion incorporating bottom friction and to a channel of varying depth. The one dimensional hydrodynamic equations for this problem are :

$$\frac{\partial U(x,t)}{\partial t} - \frac{k(x)\,U(x,t)}{\rho\,H(x)} + g\,\frac{\partial \xi(x,t)}{\partial x} = 0 \qquad (39)$$

and

$$\frac{\partial \xi(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[H(x)\,U(x,t)\right] = 0 \qquad (40)$$

where ξ, g are as defined previously, U is the depth mean current given by :

$$U = \frac{1}{H(x)}\int_{-H(x)}^{0} u(z)\,dz \qquad (41)$$

ρ the density of fluid assumed constant, $H(x)$ the depth of fluid, varying with position x , and $k(x)$ a friction factor - defining bottom stress as kU , which for complete generality also varies with position x .

Defining $K(x)$ as :

$$K(x) = \frac{k(x)}{\rho\,H(x)} \qquad (42)$$

equations (39) and (40) become

$$\frac{\partial U}{\partial t}(x,t) - K(x)\,U(x,t) + g\,\frac{\partial \xi}{\partial x}(x,t) = 0 \tag{43}$$

and

$$\frac{\partial \xi}{\partial t}(x,t) + H(x)\,\frac{\partial U}{\partial x}(x,t) + U(x,t)\,\frac{\partial H}{\partial x}(x) = 0 \tag{44}$$

The solution of equations (43) and (44) is analogous to that described previously for equations (1) and (2). Expanding $U(x,t)$ and $\xi(x,t)$ in terms of p spline functions gives :

$$U(x,t) = \sum_{i=1}^{p} \alpha_i(t)\, M_i(x) \tag{45}$$

and

$$\xi(x,t) = \sum_{i=1}^{p} \beta_i(t)\, M_i(x) \tag{46}$$

The functions $H(x)$ and $K(x)$ may also be written in terms of p splines :

$$H(x) = \sum_{i=1}^{p} a_i\, M_i(x) \tag{47}$$

$$K(x) = \sum_{i=1}^{p} b_i\, M_i(x) \tag{48}$$

where a_i , b_i are time independent coefficients.

When both $H(x)$ and $K(x)$ have known functional forms, the a_i and b_i may be determined from

$$\sum_{i=1}^{p} a_i \int_0^L M_i M_j \, dx = \int_0^L H(x) M_j \, dx \tag{49a}$$

and

$$\sum_{i=1}^{p} b_i \int_0^{L} M_i M_j \, dx = \frac{1}{\rho} \int_0 \frac{k(x)}{H(x)} M_j \, dx \tag{49b}$$

$$j = 1, 2, \ldots p$$

The integrals involving only B-splines are determined analytically, the other integrals in general requiring numerical integration.

When $H(x)$ is only available at a number of discrete points, e.g. in the form of depths read from a chart, then the coefficients a_i and b_i may be calculated in a least squares manner. The least squares technique is also used if both the depth distribution and the friction coefficient are only available in numerical form, although if the depth distribution has a functional form, the a_i can be calculated from (49a).

Equations (49) can be solved in an analogous manner to equations (9) and (10), by inversion of the matrix of integrals of spline functions.

Substituting equations (45), (46), (47) and (48) into equations (43) and (44) and applying the Galerkin method gives :

$$\int_0^L \left(\sum_{i=1}^{p} \frac{d\alpha_i}{dt} M_i(x) - \sum_{j=1}^{p} b_j M_j(x) . \sum_{i=1}^{p} \alpha_i M_i(x) + g \sum_{i=1}^{p} \beta_i \frac{dM_i}{dx}(x) \right) M_k(x) dx = 0 \tag{50}$$

and

$$\int_0^L \left(\sum_{i=1}^{p} \frac{d\beta_i}{dt} M_i(x) + \sum_{j=1}^{p} a_j M_j(x) \cdot \sum_{i=1}^{p} \alpha_i \frac{dM_i}{dx}(x) + \sum_{i=1}^{p} \alpha_i M_i(x) \cdot \sum_{j=1}^{p} a_i \frac{dM_i}{dx}(x) \right) M_k(x)\, dx = 0 \tag{52}$$

$$k = 1, 2, \ldots p$$

Due to the spatial variation of the term $K(x)$ and depth $H(x)$, two new types of integral are involved which did not occur previously, namely :

$$\int_0^L M_j(x) M_i(x) M_k(x)\, dx \quad \text{and} \quad \int_0^L M_j(x) \frac{dM_i(x)}{dx} M_k(x)\, dx$$

Expanding the B-splines in these integrals in terms of Chebyshev polynomials, yields new integrals of the form

$$\frac{\lambda_{j+1} - \lambda_j}{2} \int_{-1}^{+1} T_r(X)\, T_q(X)\, T_s(X)\, dX \quad , \quad \int_{-1}^{+1} T_r(X) \frac{dT_q}{dX}(X)\, T_s(X)\, dX$$

Since the product of two Chebyshev polynomials can be represented as the sum of two polynomials using (22), the above integrals can be written in terms of sums of integrals given in (21).

The solution of the set of coupled differential equations given by (50) and (51) can be accomplished in the manner described previously.

The Galerkin method with a basis of B-splines can also be applied to the solution of the non-linear hydrodynamic equations :

$$\frac{\partial U}{\partial t}(x,t) + U(x,t)\frac{\partial U}{\partial x}(x,t) - k(x)\frac{U(x,t)\left|U(x,t)\right|}{H(x)+\xi(x,t)} + g\frac{\partial \xi}{\partial x}(x,t) = 0 \quad (53)$$

and

$$\frac{\partial \xi}{\partial t}(x,t) + \left[H(x)+\xi(x,t)\right]\frac{\partial U}{\partial x}(x,t) + U(x,t)\frac{\partial}{\partial x}\left[H(x)+\xi(x,t)\right] = 0 \quad (54)$$

where U is now given by :

$$U(x,t) = \frac{1}{H(x)+\xi(x,t)}\int_{-H(x)}^{\xi(x,t)} u(z)\,dz \quad (55)$$

The term $k\,U|U|$ is particularly difficult to handle analytically, and, like Cartwright [16] and Doodson [17] , it is convenient to approximate it in the following manner :

$$k\,U|U| \approx k_1 U + k_2 U^3 \quad (56)$$

where k , k_1 and k_2 in general are functions of x .

Since the frictional coefficient k should vary with current strength this approximation is probably as physically realistic as retaining the $U|U|$ form. The k_1 and k_2 are chosen to minimize in a least squares sense the difference between the two expressions for a finite range of values of U .

The term $k(x)/\left[H(x)+\xi(x,t)\right]$ is initially approximated as follows :

$$\frac{k(x)}{\left[H(x)+\xi(x,t)\right]} \approx \frac{k(x)}{H(x)} \quad (57)$$

- reasonably accurate in deep water ($H(x) \gg \xi(x)$).

Incorporating these approximations in (53) gives :

$$\frac{\partial U}{\partial t}(x,t) + U(x,t)\frac{\partial U}{\partial x}(x,t) - K_1(x)U(x,t) - K_2(x)U(x,t)^3 + g\frac{\partial \xi}{\partial x}(x,t) = 0 \tag{58}$$

where $K_1(x) = \dfrac{k_1(x)}{H(x)}$ $\qquad K_2(x) = \dfrac{k_2(x)}{H(x)}$

Expanding :

$$K_1(x) = \sum_{i=1}^{p} b_i M_i(x) \tag{59}$$

$$K_2(x) = \sum_{i=1}^{p} c_i M_i(x) \tag{60}$$

and then substituting (45), (46), (47), (59), (60) into (54) and (58), using the Galerkin method, gives :

$$\int_0^L \left\{ \frac{d\beta_i}{dt} M_i(x) + \left(\sum_{i=1}^{p} a_i M_i(x) + \sum_{i=1}^{p} \beta_i M_i(x) \right) \cdot \sum_{j=1}^{p} \alpha_j \frac{dM_j}{dx}(x) \right.$$

$$\left. + \sum_{j=1}^{p} \alpha_j M_j(x) \cdot \left(\sum_{i=1}^{p} a_i \frac{dM_i}{dx}(x) + \sum_{i=1}^{p} \beta_i \frac{dM_i}{dx}(x) \right) \right\} M_k(x)\,dx = 0 \tag{61}$$

$$k = 1, 2, \ldots, p$$

and

$$\int_0^L \left[\left\{ \sum_{i=1}^{p} \frac{d\alpha_i}{dt} M_i(x) + \sum_{i=1}^{p} \alpha_i M_i(x) . \sum_{j=1}^{p} \alpha_j \frac{dM_j}{dx}(x) \right. \right.$$

$$- \sum_{i=1}^{p} b_i M_i(x) . \sum_{j=1}^{p} \alpha_j M_j(x) - \sum_{i=1}^{p} c_i M_i(x) . \left(\sum_{j=1}^{p} \alpha_j M_j(x) \right)^3$$

$$\left. + g \sum_{i=1}^{p} \beta_i \frac{dM_i}{dx} \right\} M_k(x) dx = 0 \qquad (62)$$

$$k = 1, 2, \ldots, p$$

The U^3 terms give rise to integrals of the form :

$$\int_0^L M_i(x) M_j(x) M_k(x) M_l(x) M_m(x) dx$$

Expanding the B-splines in terms of Chebyshev polynomials, and using (22), this integral can be expressed as a sum of integrals involving products of two Chebyshev polynomials.

If $\xi(x, t)$ is retained in the denominator of the friction term, then

$$K_1(x,t) = \frac{k_1(x)}{H(x) + \xi(x,t)} , \quad K_2(x,t) = \frac{k_2(x)}{H(x) + \xi(x,t)} \qquad (63)$$

In solving numerically (53) and (54) it is usual to represent time derivatives, by a forward difference formula. Using this discretisation in time, the time dependent functions K_1 and K_2 may be readily incorporated, the coefficients b_i and c_i

being recalculated at each time step using values of ξ at the lower time level, prior to solving (62) to step forward in time.

The calculation of the b_i and c_i involving the solution of :

$$\sum_{i=1}^{p} b_i \int_0^L M_i M_j \, dx = \int_0^L \frac{k_1(x)}{H(x)+\xi(x,t)} M_j \, dx \tag{64a}$$

and

$$\sum_{i=1}^{p} c_i \int_0^L M_i M_j \, dx = \int_0^L \frac{k_2(x)}{H(x)+\xi(x,t)} M_j \, dx \tag{64b}$$

$$j = 1, 2, \ldots, p$$

at each time step.

Retaining the term $U|U|$ in equation (53), and using the Galerkin procedure, as previously, requires the evaluation of the integral :

$$\int_0^L \left\{ \frac{k(x) \sum_{i=1}^{p} \alpha_i M_i(x) \left| \sum_{i=1}^{p} \alpha_i M_i(x) \right|}{H(x) + \sum_{i=1}^{p} \beta_i M_i(x)} \right\} M_k(x) \, dx \tag{65}$$

which it is possible to compute numerically at each time step using the α_i and β_i determined at the lower time level.

A number of partial differential equations containing both non-linear and dispersive terms exist in the literature, having analytical solutions with which numerical results can be compared. Burgers equation serves as an excellent model for the non-linear hydrodynamic equations, and comparing results obtained using an expansion of B-splines with analytical results, Davies [18] shows that highly accurate solutions can be obtained, for non-linear problems.

The method given here involving B-splines has also been used to calculate the vertical profile of horizontal current in terms of an expansion of B-splines [19,20] , good agreement being attained with results obtained by Heaps [21] , confirming the accuracy and general applicability of the method.

6. CONCLUSIONS

The results presented here demonstrate how readily various hydrodynamic problems in Oceanography, with boundary conditions incorporated, can be solved, using a basis of B-splines. Using the recurrence algorithm to generate these splines allows their order to be readily changed; the multiplicity and positions of the knots is completely flexible.

The method yields accurate results for all the boundary conditions considered, the accuracy being improved by increasing both the number of basis functions and the order of the spline.

The extension of the technique to problems involving non-linear terms with ability to deal with sea bed topography and spatially varying friction, and the accuracy of solution obtainable, illustrate the power of the method.

ACKNOWLEDGEMENTS

The author is indebted to Dr. N. S. Heaps for his help and encouragement, and also for a number of valuable suggestions.

REFERENCES

1. GALERKIN, B.G. 1915 Rods and plates. Series occurring in various questions concerning the elastic equilibrium of rods and plates. Engineers Bulletin, 19, 897-908.

2. ORSZAG, S.A. 1971 Numerical simulation of incompressible flows within simple boundaries : accuracy. Journal of Fluid Mechanics, 49, 75-112.

3. ORSZAG, S.A. 1971 Numerical simulation of incompressible flows within simple boundaries. (1)Galerkin (Spectral) representations. Studies in Applied Mathematics, 4, 293-327.

4. CULLEN, M.J.P. 1974 Integrations of the primitive equations on a sphere using the finite element method. Quarterly Journal of the Royal Meteorological Society, 100, 555-562.

5. WANG, H.H., HALPEN, P., DOUGLAS, J. and DUPONT, T. 1972 Numerical solutions of the one-dimensional primitive equations using Galerkin approximations with localized basis functions. Monthly Weather Review, 100, 738-746.

6. RAO, D.B. 1973 Spectral methods of forecasting storm surges. Hydrological Sciences Bulletin. 18, 311-316.

7. GROTKOP, G. 1973 Finite element analysis of long-period water waves. Computer Methods in Applied Mechanics and Engineering, 2, 147-157.

8. SCHOENBERG, I.J. 1946 Contributions to the problem of approximation of equidistant data by analytic functions. Quarterly Journal of Applied Mathematics, 4, 45-99.

9. POWELL, M.J.D. 1970 Curve fitting by splines in one variable. pp 65-83, in Numerical approximations to functions and data. (ed. J. G. Hayes). London: Athlone Press. 177 pp.

10. SCHUMAKER, L.L. 1969 Some algorithms for the computation of interpolating and approximating spline functions, pp 87-102, in _Theory and application of spline functions_, (ed. T. N. E. Greville). New York: Academic Press. 212 pp.

11. AHLBERG, J.H., NILSON, E.N., and WALSH, J.L. 1967 _The theory of splines and their applications_. New York: Academic Press.

12. COX, M.G. 1972 The numerical evaluation of B-splines. _Journal of the Institute of Mathematics and Applications_. _10_ 134-149.

13. FOX, L. and PARKER, I.B. 1968 _Chebyshev polynomials in numerical analysis_. Oxford University Press, pp 205.

14. KREISS, H.O. and OLIGER, J. 1972 Comparison of accurate methods for the integration of hyperbolic equations. _Tellus_ _24_ 199-215.

15. BROWNING, G., KREISS, H.O., and OLIGER, J. 1973 Mesh Refinement. _Mathematics of Computation_ _27_, 29-39.

16. CARTWRIGHT, D.E. 1968 A unified analysis of tides and surges round north and east Britain. _Philosophical Transactions of the Royal Society_, _A_, _263_, 1-55.

17. DOODSON, A.T. 1956 Tides and storm surges in a long uniform gulf. _Proceedings of the Royal Society_, _A_, _237_, 325-343.

18. DAVIES, A.M. 1976 A numerical investigation of errors arising in applying the Galerkin method to the solution of non-linear partial differential equations submitted to _Computer Methods in Applied Mechanics and Engineering_.

19. DAVIES, A.M. 1976 The numerical solution of the three-dimensional hydrodynamic equations, using a B-spline representation of the vertical current profile. _Mémoires de la Société Royale des Sciences de Liège_ - in press.

20. DAVIES, A.M. 1976 Three dimensional model with depth-varying eddy viscosity. _Mémoires de la Société Royale des Sciences de Liège_ - in press.

21. HEAPS, N.S. 1971 On the numerical solution of the three-dimensional hydrodynamical equations for tides and storm surges. _Mémoires de la Société Royale des Sciences de Liège_, ser 6, 2, 143-180.

TABLE I

Errors using 4th order B-splines (τ = 1.0), with a periodic boundary condition

Time (Hrs)		6 basis functions s.p.	d.p.	12 basis functions s.p.	d.p.	18 basis functions s.p.	d.p.
		(a)	(b)				
10	$\Delta\xi$	3.97E-4	3.90D-4	3.31E-5	2.01D-5	4.68E-5	4.13D-6
	ΔU	2.16E-3	1.96D-3	1.65E-4	1.85D-4	1.17E-4	9.38D-5
20	$\Delta\xi$	4.65E-4	3.91D-4	4.69E-5	2.01D-5	8.86E-5	5.35D-6
	ΔU	2.40E-3	1.98D-3	2.25E-4	2.05D-4	2.23E-4	1.69D-4
30	$\Delta\xi$	5.51E-4	3.92D-4	6.38E-5	2.00D-5	1.35E-4	7.70D-6
	ΔU	2.66E-3	2.01D-3	2.88E-4	2.53D-4	3.55E-4	2.51D-4
40	$\Delta\xi$	6.37E-4	3.95D-4	8.73E-5	2.01D-5	1.81E-4	1.01D-5
	ΔU	2.91E-3	2.05D-3	3.48E-4	3.02D-4	4.60E-4	3.37D-4
50	$\Delta\xi$	7.21E-4	3.99D-4	1.03E-4	2.01D-5	2.29E-4	1.25D-5
	ΔU	3.17E-3	2.11D-3	4.21E-4	3.50D-4	5.75E-4	4.27D-4
60	$\Delta\xi$	8.40E-4	4.02D-4	1.20E-4	2.02D-5	2.75E-4	1.49D-5
	ΔU	3.42E-3	2.16D-3	4.91E-4	3.99D-4	7.07E-4	5.18D-4

(a) 3.97E-4 = 3.97×10^{-4} E indicating single precision

(b) 3.90D-4 = 3.90×10^{-4} D indicating double precision

TABLE II

Errors using 6th order B-splines ($\tau = 1.0$) with a periodic boundary condition

Time (Hrs)		6 basis functions s.p.	6 basis functions d.p.	12 basis functions s.p.	12 basis functions d.p.
10	$\Delta\xi$	1.07E-4	1.35D-5	2.03E-5	3.45D-6
	Δu	1.57E-4	1.16D-4	1.74E-4	9.29D-5
20	$\Delta\xi$	1.82E-4	1.35D-5	3.57E-5	6.47D-6
	Δu	2.62E-4	1.60D-4	3.16E-4	1.91D-4
30	$\Delta\xi$	2.61E-4	1.35D-5	5.03E-5	9.49D-6
	Δu	3.72E-4	2.05D-4	4.85E-4	2.95D-4
40	$\Delta\xi$	3.40E-4	1.42D-5	6.71E-5	1.25D-5
	Δu	4.73E-4	2.53D-4	6.50E-4	4.05D-4
50	$\Delta\xi$	4.19E-4	1.56D-5	8.35E-5	1.56D-5
	Δu	5.59E-4	3.02D-4	8.19E-4	5.16D-4
60	$\Delta\xi$	4.91E-4	1.67D-5	1.00E-4	1.86D-5
	Δu	6.48E-4	3.52D-4	9.96E-4	6.30D-4

TABLE III

Errors using 4th and 6th order B-splines ($\tau = 2.0$)
with a periodic boundary condition

Time (Hrs)		4th order spline Number of basis functions 6	10	12	6th order spline Number of basis functions 6	10	12
10	$\Delta\xi$	1.46D-2	4.55D-4	1.93D-4	1.63D-3	3.14D-5	2.10D-5
	ΔU	6.57D-2	2.42D-3	1.04D-3	8.55D-3	2.45D-4	1.94D-4
20	$\Delta\xi$	2.78D-2	4.83D-4	1.99D-4	2.28D-3	5.23D-5	4.04D-5
	ΔU	1.38D-1	2.64D-3	1.17D-3	1.09D-2	3.97D-4	3.88D-4
30	$\Delta\xi$	4.13D-2	5.31D-4	2.11D-4	2.95D-3	7.09D-5	6.03D-5
	ΔU	2.11D-1	2.95D-3	1.31D-3	1.46D-2	5.64D-4	5.84D-4
40	$\Delta\xi$	5.46D-2	5.98D-4	2.29D-4	3.72D-3	9.04D-5	8.08D-5
	ΔU	2.80D-1	3.33D-3	1.49D-3	1.96D-2	7.45D-4	7.95D-4
50	$\Delta\xi$	6.77D-2	6.76D-4	2.50D-4	4.52D-3	1.10D-4	1.01D-4
	ΔU	3.66D-1	3.75D-3	1.68D-3	2.42D-2	9.73D-4	1.03D-3
60	$\Delta\xi$	8.01D-2	7.59D-4	2.75D-4	5.35D-3	1.30D-4	1.21D-4
	ΔU	4.36D-1	4.29D-3	1.92D-3	2.91D-2	1.15D-3	1.23D-3

TABLE IV

Variation of errors with time for problem (b)(ii) using 4th order B-splines

Time Cycles		Number of basis functions 5	10	15
3	$\Delta\xi$	4.92D-1	4.66D-3	5.85D-4
6	$\Delta\xi$	5.23D-1	4.63D-3	5.61D-4
9	$\Delta\xi$	4.72D-1	4.27D-3	5.71D-4
12	$\Delta\xi$	4.92D-1	4.80D-3	5.66D-4
15	$\Delta\xi$	5.21D-1	4.60D-3	5.62D-4

TABLE V

Errors after 15 cycles for problem (b), using a number of wavelengths (cases (i), (ii) and (iii))

Case		4th order spline Number of basis functions			6th order spline Number of basis functions		
		5	10	15	5	10	15
(i)	$\Delta\xi$	3.91D-3	1.59D-4	2.75D-5	1.69D-4	1.01D-5	1.64D-5
(ii)	$\Delta\xi$	5.21D-1	4.60D-3	5.62D-4	5.69D-2	2.30D-4	2.11D-5
(iii)	$\Delta\xi$	1.27D 0	2.56D-1	2.98D-2	1.06D 0	3.62D-2	1.82D-3

TABLE VI

Errors after 15 cycles for problem (c), cases (i) and (ii)

Case		4th order B-spline Number of basis functions 6	10	12	6th order B-spline Number of basis functions 6	10	12
(i)	$\Delta\xi$	7.09D-3	4.32D-4	1.76D-4	2.98D-4	2.66D-4	1.57D-4
	ΔU	3.47D-3	1.99D-4	7.64D-5	1.07D-4	1.02D-4	5.70D-5
(ii)	$\Delta\xi$	1.10D 0	7.74D-2	2.29D-2	3.01D-1	9.49D-4	4.79D-4
	ΔU	2.96D-1	2.63D-2	8.03D-3	1.04D-2	3.94D-4	1.99D-4

TABLE VII

Errors after 15 cycles for problem (d), cases (i) and (ii)

Case		4th order B-spline Number of basis functions 5	7	10	6th order B-spline Number of basis functions 5	7	10
(i)	$\Delta\xi$	4.14D-3	7.09D-4	5.56D-5	5.26D-5	4.41D-5	4.26D-5
	ΔU	2.51D-4	6.45D-5	1.95D-5	1.38D-5	1.32D-5	1.31D-5
(ii)	$\Delta\xi$	2.63D-2	4.98D-3	3.30D-3	2.69D-3	1.97D-3	1.94D-3
	ΔU	9.64D-3	2.16D-3	7.20D-4	7.17D-4	6.58D-4	6.55D-4