A wave-averaged energy equation: Comment on "Global Estimates of Wind Energy Input to Subinertial Motions in the Ekman-Stokes Layer" by Bin Liu, Kejian Wu and Changlong Guan

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In a recent paper, Liu *et al.* (2007) formulate an expression for how surface gravity waves modify the Ekman layer energy budget. They then diagnose the effect in the world oceans using available data. This comment addresses the formulation of the energy equation that is fundamental to their study.

1 Introduction

Stokes drift $\boldsymbol{U_s}$ is a Lagrangian velocity associated with linear surface gravity waves (Stokes, 1847) that are averaged over a period much greater than the wave orbital period. Stokes drift is a small correction (second order in waveslope) to the Eulerian velocity and is the result of nonlinear advection of momentum by wave orbital velocities (Phillips, 1977). For a frame of reference that is rotating it has long been known that Stokes drift can exert a force on a flow (Hasselmann, 1970). Defining the upward unit vector $\hat{\mathbf{z}}$, Coriolis parameter f, and $\boldsymbol{f} = f\hat{\mathbf{z}}$, then this wave averaged force $\boldsymbol{f} \times \boldsymbol{U_s}$, referred to as the Hasselmann force or the Stokes-Coriolis force, can be incorporated into a simple steady Ekman layer model for mean horizontal flow. Though this force does not affect the net transport of fluid, it does redistribute the momentum throughout the depth of the Ekman layer (see for example Polton *et al.*, 2005, and references therein).

Incorporating the Stokes-Coriolis force into an energy balance for the Ekman layer should be done with caution. Liu *et al.* (2007) formulate an expression for energy by taking the scalar product of wave-averaged momentum with wave-averaged velocity. This however neglects the fact that velocity and the non linear momentum term, which when wave averaged results in $\boldsymbol{f} \times \boldsymbol{U_s}$, have correlating wave varying parts. When wave averaged these correlated terms give rise to an additional contribution to the energy budget, which is a function of Stokes drift and neglected in the Liu *et al.* (2007) study. In addition to being more complete, the modified energy term presented here could in principle be entirely calculated from remotely sensed observational data, dispensing with uncertainties in modelling the vertical structure of momentum in the Ekman layer.

To make this subtlety in wave averaging clear, the process is demonstrated first for momentum, giving the familiar Coriolis-Stokes force, and then repeated for energy.

2 Wave averaging the momentum budget

For clarity a simplified ocean scenario is considered and the effects of rotation on a steady monochromatic surface gravity wave in the deep water limit is investigated. Without loss of generality the wave displacement of the air-sea interface is given the form

$$\tilde{\eta} = a\cos(kx - \sigma t) \tag{1}$$

with amplitude a, wavenumber k and frequency σ. For constant density ρ, the governing equations for the wave velocity $\tilde{\boldsymbol{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$ are

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + \boldsymbol{f} \times \tilde{\boldsymbol{u}} = -\frac{1}{\rho}\nabla\tilde{p} - g\hat{\mathbf{z}} \tag{2}$$

$$\nabla \cdot \tilde{\boldsymbol{u}} = 0 \tag{3}$$

subject to the boundary conditions

$$\tilde{w} \rightarrow 0 \quad \text{as } z \rightarrow -\infty \tag{4}$$

$$\tilde{p} = 0 \quad \text{at } z = \tilde{\eta} \tag{5}$$

Following Weber (1990) and Xu and Bowen (1994) an exact expression for the inviscid plane wave solution is given by

$$\tilde{\boldsymbol{u}} = a\sigma e^{\lambda z}\Big(\frac{\lambda}{k}\cos(kx-\sigma t), \frac{f}{\sigma}\frac{\lambda}{k}sin(kx-\sigma t), \sin(kx-\sigma t)\Big) \tag{6}$$

and

$$\tilde{p} = \rho g a e^{\lambda z}\cos\left(kx-\sigma t\right) - \rho g z \tag{7}$$

where $\lambda = k/\sqrt{1-(f/\sigma)^2}$.

Hence the wave stress $<\tilde{u}\tilde{w}>= 0$, where angle brackets denote averaging over many wave periods. However, planetary rotation gives rise to a horizontal component of wave orbital velocity at right angles to the wave propagation direction such that $<\tilde{v}\tilde{w}>\neq 0$. In particular, for surface gravity waves $f/\sigma \ll 1$,

$$\frac{\partial}{\partial z} < \tilde{v}\tilde{w} >= f a^2 k\sigma e^{2kz} \equiv f U_s. \tag{8}$$

Thus, an incompressible Eulerian flow, $\boldsymbol{u}$, that can be decomposed into a steady horizontally homogeneous part, $\boldsymbol{U}(z)$, and a wave component, $\tilde{\boldsymbol{u}}$, such that

$$\boldsymbol{u} = \boldsymbol{U}(z) + \tilde{\boldsymbol{u}}, \tag{9}$$

will have a non zero wave contribution in the wave averaged momentum budget. This contribution arises from the advection of momentum:

$$< \boldsymbol{u}\cdot\nabla\boldsymbol{u} >=< \tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}} >= \boldsymbol{f}\times\boldsymbol{U_s}. \tag{10}$$

Hence, for some stress $\boldsymbol{\tau}$ and pressure p the momentum budget,

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \rho\boldsymbol{f}\times\boldsymbol{u} = -\nabla p + \frac{\partial \boldsymbol{\tau}}{\partial z} - g\rho\hat{\mathbf{z}}, \tag{11}$$

results in the wave averaged Ekman layer momentum equations (Liu *et al.*, 2007, see for example equation 1):

$$\rho\frac{\partial \boldsymbol{U}}{\partial t} + \rho\boldsymbol{f}\times(\boldsymbol{U}+\boldsymbol{U_s}) = \frac{\partial \boldsymbol{\tau}}{\partial z}. \tag{12}$$

3 Wave averaging the energy budget

Similarly an Ekman layer wave-averaged energy equation can be obtained from (11). It is essential that the energy equation is first obtained before wave averaging is applied. This gives:

$$\rho\frac{\partial}{\partial t}\frac{<\boldsymbol{u}^2>}{2} + \rho<\boldsymbol{u}\cdot\nabla\frac{\boldsymbol{u}^2}{2}> = -<\boldsymbol{u}\cdot\nabla p> + <\frac{\partial \boldsymbol{\tau}}{\partial z}\cdot\boldsymbol{u}>. \tag{13}$$

For a steady wave field

$$\frac{\partial}{\partial t}\frac{<\boldsymbol{u}^2>}{2} = \frac{\partial}{\partial t}\frac{\boldsymbol{U}^2}{2}. \tag{14}$$

In the absence of external pressure gradients, $p = \tilde{p}$, the wave averaged pressure work term reduces to zero. If, also following Liu *et al.* (2007), the stress can be parameterised by the mean velocity shear and a coefficient of eddy viscosity K, such that

$$\boldsymbol{\tau}(z) = K \frac{\partial}{\partial z} \boldsymbol{U}(z), \tag{15}$$

then the stress term in (13) reduces to $\frac{\partial}{\partial z} < \boldsymbol{\tau} \cdot \boldsymbol{U} > -\rho < K |\frac{\partial \boldsymbol{U}}{\partial z}|^2 >$.

In the following, depth integrated quantities are computed. Of particular interest is the depth integrated Stokes-Coriolis term (8) and its manifestation in the energy budget. However, this term is complicated at depths between the wave peaks and wave troughs since the wave averaging in (8) is defined for a single phase fluid only. To avoid this problem we integrate only up to the base of the troughs, and for convenience relabel the depth to be $z = 0$ there. If we assume that near the surface the mean velocity field varies sufficiently slowly with depth, relative to the wave velocity, then relabelling the zero depth location makes a negligible difference to terms involving the mean flow. Similarly the Stokes drift terms can be trivially rescaled to accommodate this change in z and the derivation for the energy budget can proceed in a way that is as close as possible to that of Liu *et al.*. For small amplitude waves this Stokes drift rescaling factor, $e^{-2ka} \approx 1$, can be neglected. Since $< . >$ and $\partial / \partial z$ are now commutative for $z < 0$ the term for advection of energy can be rearranged to give

$$\begin{aligned}
< \boldsymbol{u} \cdot \nabla \frac{\boldsymbol{u}^2}{2} > &= < \frac{\partial}{\partial z}(\tilde{w}\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}) >, &(16)\\
&= \frac{\partial}{\partial z}(< \tilde{w}\tilde{\boldsymbol{u}} > \cdot \boldsymbol{U}), &(17)\\
&= \frac{\partial}{\partial z}[\boldsymbol{f} \times \boldsymbol{T_s} \cdot \boldsymbol{U}] &(18)
\end{aligned}$$

where $\boldsymbol{T_s}(z)$ is the Stokes transport, given by,

$$\boldsymbol{T_s}(z) = \int_{-\infty}^{z} \boldsymbol{U_s}(z)\, dz. \tag{19}$$

Assuming that the stress $\boldsymbol{\tau}(z \to -\infty) = 0$ then the depth integrated, wave averaged energy equation is

$$\frac{\partial E}{\partial t} = E_w + E_s - D, \tag{20}$$

where

$$\begin{aligned}
E &= \int_{-\infty}^{0} \frac{\rho}{2}\boldsymbol{U}^2 dz, &(21)\\
E_w &= \boldsymbol{\tau}(0) \cdot \boldsymbol{U}(0), &(22)\\
E_s &= -\rho \boldsymbol{f} \times \boldsymbol{T_s}(0) \cdot \boldsymbol{U}(0), &(23)\\
D &= \int_{-\infty}^{0} \rho K \Big|\frac{\partial \boldsymbol{U}}{\partial z}\Big|^2 dz. &(24)
\end{aligned}$$

This can be compared with the Liu *et al.* (2007, equation (6)) expression for 'total energy' in the Ekman layer. The difference is in the Stokes drift contribution, E_s, and is accounted for by the order in which the wave averaging and the scalar product are taken. Expanding the advection of energy term (18) and integrating over depth gives

$$\boldsymbol{f} \times \boldsymbol{T_s}(0) \cdot \boldsymbol{U}(0) = \int_{-\infty}^{0} \boldsymbol{f} \times \boldsymbol{U_s} \cdot \boldsymbol{U} dz + \int_{-\infty}^{0} \boldsymbol{f} \times \boldsymbol{T_s} \cdot \frac{\partial \boldsymbol{U}}{\partial z} dz. \qquad (25)$$

The first term on the right hand side is the scalar product contribution from wave-averaged momentum and wave-averaged velocity and hence gives the Liu *et al.* expression for E_s. The second term on the right hand side arises from correlations between the wave varying components of the same terms.

Computing the total Stokes drift contribution to the depth integrated energy budget (23) offers considerably advantages over evaluating a portion of the total. Besides the obvious advantage in having a complete expression for the contribution to the energy budget from Stokes drift, the total contribution is a function of surface values only whereas the component terms require a knowledge of the depth varying Ekman velocity. Modelling the vertical profile of Ekman velocities with any degree of accuracy is notoriously difficult largely because of a lack of corroborating observational data to determine the eddy viscosity profile (Huang, 1979; Price and Sundermeyer, 1999; Briscoe and Weller, 1984; Price *et al.*, 1987; Chereskin, 1995; Lenn and Chereskin, 2008). Consequently this approach is worth avoiding.

It is hard to foresee how this reformulation of the energy budget will affect the findings reported in Liu *et al.* (2007) since the difference includes a non trivial function of the vertical variation of velocity with depth. However, the modifications to their energy budget are given in the appendix.

As a final note is should be borne in mind that a number of key assumptions have been made in order to construct this model. None of these assumptions present critical flaws in modelling the real ocean, though it is of value to make these limitations explicit: Only the energy budget below the troughs is considered; it is assumed that there are no horizontal pressure gradients that are not attributed to the waves, and that the density is constant; it is assumed that, despite any energy that the waves may impart to the mean flow, the waves are maintained in a statistically steady state. Finally it is assumed that the waves are inviscid and interactions between the waves and turbulence can be neglected.

Appendix

Accepting the above assumptions then, following Liu *et al.* (2007), the surface velocity $\boldsymbol{U}(0)$ can be solved for as the solution to the steady Ekman problem, with constant eddy viscosity. Adopting their notation and expressions for F_1 and F_2, Ekman depth scale d_e. and ratio of Ekman depth to Stokes depth scales $c = d_e.2k$, the Stokes drift contribution to the energy budget (shown here only for the northern hemisphere) can be written as

$$E_s = \frac{1}{c}\Big[\boldsymbol{\tau}(0)\cdot\boldsymbol{U_s}(0) + \boldsymbol{\tau}(0)\times\boldsymbol{U_s}(0)\cdot\hat{\mathbf{z}} + \rho f d_e|\boldsymbol{U_s}(0)|^2 F_2(c)\Big]. \tag{26}$$

The total change in energy from the combined effects of wind and waves, E_{tot} is given by

$$\begin{aligned} E_{tot} &= E_w + E_s && (27)\\ &= \frac{|\boldsymbol{\tau}(0)|^2}{\rho d_e f} \quad + \quad \rho f d_e|\boldsymbol{U_s}(0)|^2 F_2(c) && (28)\\ &\quad + \quad \boldsymbol{\tau}(0)\cdot\boldsymbol{U_s}(0)(1/c - F_1(c)) && (29)\\ &\quad + \quad \boldsymbol{\tau}(0)\times\boldsymbol{U_s}(0)\cdot\hat{\mathbf{z}}(1/c + F_2(c)). && (30)\end{aligned}$$

where

$$\begin{aligned} F_1(x) &= \frac{x+2}{(x+1)^2+1}, && (31)\\ F_2(x) &= \frac{x}{(x+1)^2+1}. && (32)\end{aligned}$$

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