Accelerating dense-water flow down a slope

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ABSTRACT

Where water is denser on a shallow shelf than in the adjacent deep ocean, it tends to flow down the slope from shelf to ocean. The flow can be in a steady bottom boundary layer for moderate combinations of up-slope density gradient $-\rho_{x\infty}$ and bottom slope (angle θ to horizontal):

$$b \equiv |\rho_{x\infty}|\, g \sin\theta \,/\, (f^2 \rho_0) < 1.$$

Here g is acceleration due to gravity, ρ_0 is a mean density and f is twice the component of earth's rotation normal to the sloping bottom. For stronger combinations of horizontal density gradient and bottom slope, the flow accelerates. Analysis of an idealised initial-value problem shows that when $b \geq 1$ there is a bottom boundary layer with down-slope flow, intensifying exponentially at a rate $fb^2(1+b)^{-1/2}/2$, and slower-growing flow higher up. For stronger stratification $b > 2^{1/2}$, i.e. relatively weak Coriolis constraint, the idealised problem posed here may not be the most apposite but suggests that the whole water column accelerates, at a rate $[\rho_0^{-1}|\rho_{x\infty}|\, g \sin\theta]^{1/2}$ if f is negligible.

1. Introduction

Dense-water flows down continental slopes are of interest for their irreversible exchange of oceanic and shelf waters and their contents, hence ventilation of intermediate and abyssal waters, affecting thermohaline circulation. In particular, down-slope flows can follow winter cooling of shallow shelf seas at mid-to-high latitudes. The process is reviewed in Shapiro *et al*. (2003); an inventory of documented occurrences is given in Ivanov *et al*. (2004).

Such "cascading" in its later stages is reasonably described as a steady frictional bottom boundary layer (Shapiro and Hill, 1997). Here the flow has components (i) along the slope, as would be implied by geostrophic balance, and (ii) down the slope, as friction breaks the geostrophic constraint and allows the denser water to "seek its own level". However, preconditioning is needed; density on the shallower shelf increases to the point where such down-slope flow accelerates (from zero initially).

Bottom-boundary layers in the presence of density gradients over a slope were analysed by Garrett, McCready and Rhines (1993); layer thickness depends on the density gradient and slope. [Thickness of a mixed bottom boundary layer developed under along-slope flow also depends on sense of flow: thinner if the flows are in an upwelling sense, thicker if in a downwelling sense (Weatherly and Martin 1978; Trowbridge and Lentz 1991). Boundary layer thickness under polarised oscillatory flow of frequency $\sigma < f$ scales as $[\nu/(f \pm \sigma)]^{1/2}$, i.e. the layer is thicker under anticyclonically-polarised flow]. Shapiro and Hill (1997) and Baines (2005) studied down-slope flows of denser fluid. Shapiro and Hill's (1997) analysis assumed homogeneous fluid above and neglected acceleration terms. Baines (2005) distinguished detraining gravity currents on small slopes from entraining plumes on greater slopes (in broad terms, without rotation; "slope" here represents a combination of slope and density difference). These previous studies focused on cases where the base-state density varied only vertically, being horizontally uniform.

Here, we are concerned with a base-state density varying primarily in the horizontal (figure 1; an up-slope density increase as heat is removed, e.g. in winter, from shallower shelf waters). In fact for horizontal gradients of density, the analysis of Garrett, McCready and Rhines (1993) still applies (after subtracting the implied geostrophically-balanced flow along the slope; density gradient is resolved as components parallel and normal to the sloping bottom). Their conclusion of layer-thickness dependence on density gradient and slope also applies. In our case (base-state density varying primarily in the horizontal) the bottom boundary layer can be arbitrarily thick as the up-slope density gradient $-\rho_{x\infty}$ and bottom slope θ (figure 1) increase to

$$|\rho_{x\infty}|g\sin\theta = \rho_0 f^2$$

where g is acceleration due to gravity, ρ_0 is a uniform mean density, f is the Coriolis parameter. Shapiro *et al*. (2003) suggested that this breakdown of a steady bottom-boundary-layer solution may be a threshold for the accelerating phase of a cascade; intuitively as the tendency of denser water to "seek its own level" strengthens (with slope and density gradient) to break the steady frictional-cum-geostrophic balance.

Here we pursue this suggestion by analysing time-development from a base state with uniform slope and up-slope density gradient, balanced geostrophically by flow initially along the slope. Relative to Baines' (2005) study, we consider an early and upper-slope stage in the flow development, viz. initial acceleration from a steady state; we omit the influence of increased oceanic density with depth, but include rotation (absent from Baines, 2005).

The (idealized) problem is formulated in section 2, and analysed in section 3 according to the strength of density gradient and slope relative to the threshold for break-down of the steady bottom-boundary-layer. Section 4 presents confirmatory simpler analysis for exponentially growing solutions specifically. The findings are discussed in section 5. To help present the argument, much detailed analysis is separated to Appendices.

2. Formulation

We consider a density gradient across a slope (figure 1, upper panel). An up-slope density increase may occur especially as a gradient of (F_Q/H), if F_Q is heat removed (e.g. in winter) from shelf water of depth H. [Usually winter convection is limited to a finite depth beyond which this cause of density gradient ceases; the figure also depicts this, an along-slope velocity (shear) in geostrophic balance with the density gradient and a corresponding surface slope].

We focus on the bottom boundary layer over a slope. Co-ordinates x, z are rotated through the bottom slope θ so that x is down-slope (figure 1, lower panel). A base-state density gradient ($\rho_{x\infty}x + \rho_{z\infty}z$) can represent any combination of uniform stratification (density variation in the vertical) and lateral density variation across the slope. The following simplifications are made: no along-slope (y) gradients ($u_y = \mathrm{v}_y = 0 = p_y$) for the simplest context; no diffusion parallel to the bottom (consistent with uniform base-state density gradient); negligible cross-slope advection of momentum in the boundary layer (linearization); Boussinesq approximation (uniform density in the acceleration and viscous terms). Equations of motion including all Coriolis components and gravity in the θ-rotated coordinates are then

$$u_t + f^y w - f^z \mathrm{v} = - p_x / \rho_0 + g\rho/\rho_0 \sin\theta + (\nu u_z)_z \quad (1a)$$

$$\mathrm{v}_t + f^z u - f^x w = (\nu \mathrm{v}_z)_z \quad (1b)$$

$$w_t + f^x \mathrm{v} - f^y u = - p_z / \rho_0 - g\rho/\rho_0 \cos\theta \quad (1c)$$

$$\rho_t + u\rho_x + w\rho_z = (K\rho_z)_z \quad (1d)$$

$$u_x + w_z = 0 \quad (1e)$$

Boundary conditions are

$\rho_x \to \rho_{x\infty}$, $\quad \rho_z \to \rho_{z\infty}$, $\quad$ as $z \to \infty$

$u, w \to 0$, $\quad \mathrm{v}_z \to - g\,\rho_H / \rho_0 f^z$ $\quad$ as $z \to \infty$ (on the boundary-layer scale $(\nu/f^z)^{1/2}$)

u, v, w, $\rho_z = 0$ $\quad$ at $z = 0$.

Here (u, v, w) are (x, y, z) components of velocity, p is total pressure, g is gravitational acceleration ~ 10 m s^{-2}, ρ (total density) includes uniform constant mean density ρ_0 ~ 10^3 kg m^{-3}, ρ_H is the base-state horizontal density gradient ($\rho_{x\infty}\cos\theta + \rho_{z\infty}\sin\theta$), the earth's rotation vector is

$(f^x, f^y, f^z)/2$ (for small slopes $f^z \sim$ Coriolis parameter of order 10^{-4} s^{-1}), ν is vertical viscosity and K is vertical diffusivity $\sim 10^{-2}$ m^2 s^{-1}. Subscripts x, z, t denote differentiation.

Given the stated approximations, (1) are standard. With stratification (vertical gradient of density) $g\rho_{x\infty} = \rho_0 N^2 \sin\theta$, $g\rho_{z\infty} = -\rho_0 N^2 \cos\theta$, (1) correspond with Garrett, McCready and Rhines (1993) except that (1) include all Coriolis components. The boundary conditions reflect a focus on the bottom boundary layer where the flow and normal gradient of density tend to zero on the bottom. Thus the system reduces to bottom Ekman-layer equations if $\theta, f^x, f^y, \rho_{x\infty}, \rho_{z\infty}$ and time-derivatives are all zero (for example, with some non-zero flow as $z \to \infty$). Also in the same sense as a bottom Ekman layer and Garrett, McCready and Rhines (1993):

(i) we seek uniformity in x as far as possible (without lateral boundary conditions), i.e. behaviour is locally rather than externally determined. [The character of the bottom layer will change at the top of the slope and where the density gradient ceases over deeper water in figure 1. Here we expect a transition region over a small distance in x scaling as $(\nu_H/f)^{1/2}$ (where ν_H is lateral eddy viscosity; e.g. Hill and Johnson, 1974)].

(ii) existence of a consistent pressure field is a constraint on imposed flow above the boundary layer.

Without loss of generality we write the solution of (1) in the form:

$$u = u_1$$

$$\mathrm{v} = \mathrm{v}_1 - g\, z\, \rho_H / (\rho_0 f^z) \qquad [\rho_H \equiv (\rho_{x\infty} \cos\theta + \rho_{z\infty} \sin\theta)]$$

$$w = w_1$$

$$p = p_1 + p_0 + g\rho_0(x \sin\theta - z\cos\theta) + g\rho_{x\infty}[x^2 \sin\theta + z^2 \cos\theta\, f^x/f^z]/2 + g\rho_{z\infty} z^2\, [f^x/f^z \sin\theta - \cos\theta]/2 - g\rho_{x\infty} xz \cos\theta$$

$$\rho = \rho_1 + \rho_0 + \rho_{x\infty} x + \rho_{z\infty} z$$

where p_0 and ρ_0 are uniform constants.

Formally this is just a transformation from (u, v, w, p, ρ) to (u_1, v_1, w_1, p_1, ρ_1). We use it because the terms *without* subscript "1" are nearly a solution for the base state. These terms geostrophically balance the base-state density gradient ρ_H, satisfying (1) exactly in the interior and homogenising all of the boundary conditions for (u_1, v_1, w_1, p_1, ρ_1) except for ρ_{1z} at $z = 0$. This base-state balance has velocity tending to zero at the bed (and uniform in x); p_0 is an arbitrary uniform pressure playing no dynamical role. This state is not unique, but arbitrary only in an additional along-slope flow v' and corresponding pressure distribution, as follows. From (1a, c) any additional geostrophic flow (v', p') must satisfy $\rho_0 f^z \mathrm{v}' = p'_x$ and $\rho_0 f^x \mathrm{v}' = -p'_z$. Hence directly $f^z p'_z + f^x p'_x = 0$, i.e. in terms of $\xi \equiv f^z x - f^x z$ and (any) η independent of ξ, p' is a function of ξ only. Cross-differentiating, v' is likewise a function of ξ only. Then $p'_\xi = \rho_0$ v'. Hence any geostrophic flow v' (additional to the base state) is a function only of the coordinate ξ normal to the direction (f^x, f^z) of rotation in the cross-slope plane.

We choose v' = 0 based on previous studies of flow over a sloping boundary; the bottom boundary layer adjusts the density field to form a "slippery boundary layer" (if as here the along-slope forcing is zero; e.g. Garrett, McCready and Rhines, 1993). Although these studies supposed a vertical base-state density gradient (rather than the horizontal gradient here), our base state corresponds with such an adjusted density field, having the associated "interior" geostrophic velocity tending to zero at the bed. Geostrophic flows found in shelf seas around "pools" of dense water tend to zero at the bed (Hill et al., 2008; the result of long-term action of friction). One can also argue on grounds of simplicity that this is the first case to solve.

Expressing (1) in terms of (u_1, v_1, w_1, p_1, ρ_1),

$$u_{1t} + f^y w_1 - f^z \mathrm{v}_1 = -p_{1x}/\rho_0 + g\rho_1/\rho_0 \sin\theta + (\nu u_{1z})_z \qquad \text{(2a)}$$

$$v_{1t} + f^z u_1 - f^x w_1 = (\nu v_{1z})_z \tag{2b}$$

$$w_{1t} + f^x v_1 - f^y u_1 = - p_{1z} / \rho_0 - g\rho_1/\rho_0 \cos\theta \tag{2c}$$

$$\rho_{1t} + u_1(\rho_{1x}+\rho_{x\infty}) + w_1(\rho_{1z}+\rho_{z\infty}) = (K\rho_{1z})_z \tag{2d}$$

$$u_{1x} + w_{1z} = 0 \tag{2e}$$

where terms in ν_z, K_z have been omitted as inessential complexities. Boundary conditions are

$$\rho_{1x} \to 0, \quad \rho_{1z} \to 0, \quad \text{as } z \to \infty$$

$$u_1,\ w_1,\ v_{1z} \to 0 \quad \text{as } z \to \infty$$

$$u_1,\ v_1,\ w_1,\ \rho_{1z} + \rho_{z\infty} = 0 \quad \text{at } z = 0.$$

The last condition on z=0 is the only reason for non-zero φ_1 (*i.e.* u_1, v_1, w_1, p_1 or ρ_1); zero normal diffusive flux of density distorts the density contours and drives bottom boundary-layer motion, as discussed in Wunsch (1970) but now with a horizontal density gradient. Hence there is no *a priori* source of motion φ_1 far from z=0, or of x-dependence in u_1, ρ_1. We take the simplest case $K = \nu$, and seek solutions with $u_{1x} = 0 = \rho_{1x}$, hence $w_1 = 0$ [from (2e) and w=0 at z=0]. Thus the non-linear terms in (2d) and the (2a,b) terms in f^y, f^x are zero. In (2c), we scale the boundary-layer depth in z by $(\nu/f^z)^{1/2}$ and p_1, ρ_1 by $\rho_0 L_x f^z v_1$, $u_1\rho_{x\infty} / f^z$ from (2a,d) respectively. Then the (2c) terms in f^x, f^y are relatively small if (i) $f^{x,y}(\nu/f^z)^{1/2}/(L_x f^z)$ is small, i.e. if the boundary layer is thin relative to the x-scale L_x, or if (ii) $f^{x,y} f^z \rho_0 /(g\, \rho_{x\infty})$ is small, i.e. if $\rho_{x\infty}/\rho_0 >> 10^{-9}$/m or a relative density difference >> 10^{-3} per 10^3km. This is assured by small slope θ for the criterion $g\rho_{x\infty}\sin\theta/(f^2\rho_0) = 1$ under investigation, except close to the equator.

Reduced equations for $\partial_t \equiv 0$ (steady state), viz. (2d) and $\int_z^\infty [\rho_{x\infty}(2b) - f^z(2d)]dz$, give u_1 and v_{1z} respectively in terms of ρ_1; then substitution in (2a) gives

$$\{\nu^2\partial_z^4 + f^2(1\text{-}b)\}\ \rho_{1z} = 0 \tag{2f}$$

where we have written f in place of f^z. We define a "gravitational" rate λ and non-dimensional b:

$$\lambda \equiv [-g\rho_{x\infty}\rho_0^{-1}\sin\theta]^{1/2}, \qquad b \equiv -g\rho_{x\infty}\sin\theta/(f^2\rho_0) = \lambda^2/f^2 > 0 \tag{2g}$$

If $b < 1$, there is a boundary-layer solution satisfying the conditions on z=0:

$$\rho_{1z} = -\tfrac{1}{2}\,\rho_{z\infty}\ [(1+i)\ e^{-(1+i)\gamma z} + (1-i)\ e^{-(1-i)\gamma z}]$$

where $\gamma = (2\nu/f)^{-1/2}(1\text{-}b)^{1/4}$, as in Shapiro *et al.* (2003).

As b approaches 1, γ decreases in the exponents $-(1\pm i)\gamma z$ of this boundary layer form, i.e. the boundary layer becomes thicker.

If $b > 1$, the complementary functions are $\exp(\pm qz)$, $\exp(\pm iqz)$ where $q^4 = f^2(b\text{-}1)/\nu^2$; there is only one exponentially-decaying solution. Hence no steady boundary-layer satisfies conditions on both ρ_{1z} and u_1 at z=0. The criterion for this breakdown of the boundary-layer solution is

$$b>1, \qquad \text{i.e. from (2g)} \qquad -\rho_{x\infty}\, g \sin\theta > \rho_0 f^2 \qquad \text{or} \qquad \lambda > f,$$

i.e. the up-slope density gradient $-\rho_{x\infty}$ and bottom slope $\sin\theta$ are sufficiently large. As in section 1, the strengthened tendency of denser water to "seek its own level" breaks the steady frictional-cum-geostrophic boundary layer balance.

3. Time-dependence

In the absence of a steady-state boundary layer when b≥1 (strong density gradient and slope), we allow for time-dependence and consider evolution from an initial state. The analysis is by Laplace transformation of the reduced equations (2) for $u_{1x} = 0 = \rho_{1x}$ (hence $w_1 = 0$) and neglected terms in f^x, f^y.

a. Laplace Transform

Form $\int_0^\infty (2)e^{-\alpha t}dt$ and use $\varphi(\xi,\alpha)$ to denote the Laplace transform $\int_0^\infty \varphi_1\, e^{-\alpha t}dt$ of $\varphi_1(z,t)$. Thus

$$-f\mathrm{v} \quad = \quad g\rho/\rho_0 \sin\theta + \alpha D^2 u \qquad (3a)$$

$$fu \quad = \quad \alpha D^2 \mathrm{v} \qquad (3b)$$

$$p_\xi \quad = \quad -(\nu/\alpha)^{1/2}\, g\rho \cos\theta \qquad (3c)$$

$$u\rho_{x\infty} \quad = \quad \alpha D^2 \rho \qquad (3d).$$

Here, by (3c), p_x is a function of x only, hence a constant which is zero in (3a). We also use

$$\xi \equiv (\alpha/\nu)^{1/2} z, \quad D^2 \equiv \partial_\xi^{\,2}\text{-}1, \quad \text{and (below)} \quad \mu \equiv f(b\text{-}1)^{1/2} \qquad (4).$$

(3a, b, d) are homogeneous through an assumption that u_1, v_1 and ρ_1 are initially zero. Any other initial state in this linear problem will "force" a corresponding particular flow, additional to the flow that we derive here for the minimal base-state "forcing" ($\rho_{z\infty} \neq 0$ on $z = 0$) of φ_1. [As shown in section 2, there is no steady-state initial form satisfying all equations and boundary conditions; the reason for this analysis].

(3c) determines p (which appears nowhere else) in terms of ρ.

From $\rho_{x\infty}$(3b) $- f$(3d) we have $\quad D^2(\mathrm{v}\,\rho_{x\infty} - f\rho) = 0$,

i.e. $$\psi \equiv \mathrm{v}\,\rho_{x\infty} - f\rho = A(\alpha)\, e^{-\xi} \qquad (5a).$$

From (3a) + μ(3b)/f we have $\quad (\alpha D^2 - \mu)(u + \mu\mathrm{v}/f) = g \sin\theta\, (f\rho_0)^{-1}\, \psi$,

i.e. $$\chi \equiv u + \mu\mathrm{v}/f = g\sin\theta\, (f\rho_0\, \mu)^{-1} A(\alpha)\, [\exp(-q_+\xi) - \exp(-\xi)] \qquad (5b).$$

From (3b) and (5b) [or from (3d), (5a) and (5b)] we have $\alpha D^2\mathrm{v} = f\chi - \mu\,\mathrm{v}$,

i.e. $$\mathrm{v} = g\sin\theta\, (2\mu^2\rho_0)^{-1} A(\alpha)\, [\exp(-q_+\xi) - 2\exp(-\xi) + \exp(-q_-\xi)] \qquad (5c)$$

where $$q_+ = +(1 + \mu/\alpha)^{1/2}, \qquad q_- = +(1 - \mu/\alpha)^{1/2} \qquad (5d).$$

(5c, b, a) in turn give

$$\mathrm{v} = g\sin\theta\, (2\,\mu^2\rho_0)^{-1} A(\alpha)\, [\exp(-q_+\xi) - 2\exp(-\xi) + \exp(-q_-\xi)] \qquad (6a)$$

$$u = g\sin\theta\, (2f\mu\rho_0)^{-1} A(\alpha)\, [\exp(-q_+\xi) - \exp(-q_-\xi)] \qquad (6b)$$

$$\rho = (2\,\mu^2/f)^{-1} A(\alpha)\, \{2\exp(-\xi) - b[\exp(-q_+\xi) + \exp(-q_-\xi)]\} \qquad (6c)$$

in terms of $A(\alpha)$, which is determined by the Laplace transformed bottom boundary condition

$$\rho_{z\infty} = -\sqrt{(\alpha^3/\nu)}\, \rho_\xi = \sqrt{(\alpha^3/\nu)}\, (2\,\mu^2/f)^{-1} A(\alpha)\, [2 - b(q_+ + q_-)] \qquad (6d).$$

If $\alpha = 0$ our notation (4) is invalid, but the system (3a-d) is the steady-state form of (2a-d), leading to (2f) with no valid solution as already noted. The outcome is a singularity in the inverse Laplace Transform (section 3b).

If $b = 1$, then μ defined by (4) is zero. Nevertheless, the procedure that leads to (5a-c) gives

$$\psi \equiv \mathrm{v}\,\rho_{x\infty} - f\rho = A(\alpha)\, e^{-\xi}$$

$$\alpha D^2 u = g\sin\theta\, (f\rho_0)^{-1}\, \psi,$$

i.e. $$u = -g\sin\theta\, (2f\alpha\rho_0)^{-1} A(\alpha)\, \xi \exp(-\xi) \qquad (7a)$$

$\alpha D^2 \mathrm{v} = fu$,

i.e. $$\mathrm{v} = g \sin\theta\, (8\alpha^2\rho_0)^{-1} A(\alpha)\, (\xi^2 + \xi) \exp(-\xi) \tag{7b}$$

hence $$\rho = - A(\alpha)\, e^{-\xi} [f^1 + (\xi^2 + \xi)f/(8\alpha^2)] \tag{7c}$$.

Here $A(\alpha)$ is determined by the Laplace transformed bottom boundary condition

$$\rho_{z\infty} = - \sqrt{(\alpha^3/\nu)}\, \rho_\xi = (f^2\alpha\nu)^{-1/2} A(\alpha)\, [f^2/8 - \alpha^2] \tag{7d}$$.

b. Inverse Laplace Transform

The inverse Laplace transform requires that we evaluate

$$(\rho_1, u_1, \mathrm{v}_1)\,(z, t) = (2\pi i)^{-1} \int_L (\rho, u, \mathrm{v})\, e^{\alpha t}\, d\alpha$$

where the integral path L is from a-iT to a+iT as $T \to \infty$ and a is sufficiently positive for validity.

1) SOLUTION FOR $b = 1$

This is evaluated in Appendix A. For large time, it is the sum of terms decaying at least as fast as $e^{-ft/\sqrt{8}}$ plus: (A.3) terms decaying at least as fast as $t^{-1/2}$; (A.1) terms (${}^a\rho_1$, ${}^a\mathrm{v}_1$) showing a diffusive-type thickening bottom boundary layer with along-slope flow only, intensifying but finite; constant-form (A.2) terms $\{{}^g\rho_1, {}^gu_1, {}^g\mathrm{v}_1\}$ with a growth factor $\exp(\alpha_p t)$ where $\alpha_p = f/\sqrt{8}$. These (A.2) terms (figure 2) represent a bottom boundary layer, including accelerating down-slope flow, decaying into the interior on the boundary-layer (ξ) scale $(8^{-1/2}f/\nu)^{-1/2}$.

2) SOLUTION FOR $b > 1$

This is evaluated in Appendix B. For large time, it is the sum of terms decaying at least as fast as $e^{-ft\sqrt{(b-1)}}$ plus: (B.1) terms which do not grow in time but do not tend to zero outside the bottom boundary layer; (B.3) terms growing with a rate close to $e^{ft\sqrt{(b-1)}}$ and again not tending to zero outside the bottom boundary layer; constant-form (B.2) terms with fastest growth, factor $\exp(\alpha_p t)$ where $\alpha_p = f\, b^2\, (1+b)^{-1/2}/2$. These (B.2) terms (figure 3 where b = 1.4, close to a maximum $\sqrt{2}$ for validity of this solution) represent a bottom boundary layer with accelerating down-slope flow, as for b = 1. However, the scale for decay into the interior, as $\exp(-\xi_-)$, is $(2\nu/f)^{1/2}(b+1)^{1/4}(2-b^2)^{-1/2}$ which becomes large as b approaches $\sqrt{2}$.

Note that these solutions for $b \geq 1$ do not include a recognisable inertial oscillation (in the terms of either Appendix A or Appendix B). There is no impulse from the initial condition to trigger an oscillation. Moreover, the imbalance forcing the additional flow φ_1 is confined to the bottom boundary layer where friction and diffusion are as important as inertia. Nevertheless, there are other "transient" motions represented by (A.1,3) (for b = 1) and (B.1,3) (for b > 1) in addition to the respective fastest-growing boundary layer forms (A.2), (B.2).

3) LARGE b

Large b corresponds to a large up-slope density gradient, or equivalently small f. Let $f = 0$. Then the Laplace-Transformed equations (3a-d) simplify to

$$0 = g\rho/\rho_0 \sin\theta + (\nu\partial_z^2 - \alpha)u \tag{8a}$$

$$0 = (\nu\partial_z^2 - \alpha)\mathrm{v} \tag{8b}$$

$$p_z = - g\rho \cos\theta \tag{8c}$$

$$u\rho_{x\infty} = (\nu\partial_z^2 - \alpha)\rho \tag{8d}$$

where again p_x is a function of x only, hence a constant which is zero in (8a).

(8b) has the only occurrence of v which is then zero from the boundary conditions after (2). (8c) determines p (which only appears here) in terms of ρ. $-\rho_{x\infty}$(8a) $\pm\ \lambda$(8d) gives

$$(\nu\partial_z^2-\alpha)\,(-\rho_{x\infty}u\pm\lambda\rho) \quad = \quad \pm\,(-\lambda)\,(-\rho_{x\infty}u\pm\lambda\rho).$$

Thus $\quad (-\rho_{x\infty}u+\lambda\rho) \;=\; U_+(\alpha)\exp(-z_-), \qquad (-\rho_{x\infty}u-\lambda\rho) \;=\; U_-(\alpha)\exp(-z_+),$

where $\quad z_\pm \;=\; z[(\alpha\pm\lambda)/\nu]^{1/2}, \qquad \alpha$ is complex, λ is real positive by (2g).

Using the boundary conditions at $z = 0$,

$$-\rho_{x\infty}u = U(\alpha)[\exp(-z_-)-\exp(-z_+)], \qquad \lambda\rho = U(\alpha)[\exp(-z_+)+\exp(-z_-)], \qquad (9a)$$

$$\rho_{z\infty} \;=\; -\,\alpha\,\rho_z \;=\; \alpha\,\nu^{-1/2}\,U[(\alpha+\lambda)^{1/2}+(\alpha-\lambda)^{1/2}]/\lambda \qquad \text{determining } U(\alpha). \qquad (9b)$$

The solution in this case is evaluated in Appendix C. For large time, it is the sum of terms (C.2) decaying at least as fast as $e^{-\lambda t}$, plus: (C.1) terms which do not grow in time but do not tend to zero outside the bottom boundary layer; (C.3) terms with a growth rate close to $e^{\lambda t}$ and near-linear increase with distance from the bottom. The form of (C.3) is shown (with arbitrary factor) in figure 4. Unlike the cases with $b < \surd 2$, this is not a boundary-layer form; its continued growth into the interior is not consistent with the focus of the analysis on the bottom boundary layer; it suggests that the down-slope acceleration is on the scale of full depth of water which should be represented in a complete analysis. [By taking $f = 0$ we have eliminated any possible inertial oscillation. However, large b implies $\lambda >> f$ so that the growth rate is rapid relative to any inertial time-dependence.]

4. Growing components.

The solutions to the initial-value problem in $b \geq 1$ have various growing components; the fastest growths are exponential. The form and growth rate of these (fastest) exponentially-growing components can be confirmed (checked) more simply by seeking solutions in the form $\varphi_1 = e^{\alpha t}\,\varphi(\xi)$. Here φ represents u, v, w, p or ρ; the growth rate α is to be found. As before, let

$$\alpha_p = f\,b^2(1+b)^{-1/2}/2, \qquad b = -\,g\,\rho_{x\infty}/(\rho_0 f^2)\sin\theta \;=\; \lambda^2/f^2, \qquad \xi = z\surd(\alpha/\nu), \qquad \mu = f\surd(b-1).$$

From (2) with $\partial_x \equiv 0$ implying $w = 0$ from (2e), we have

$$\alpha u - f\mathrm{v} = \; g\rho/\rho_0\sin\theta \quad + \alpha u_{\xi\xi} \qquad (10a)$$

$$\alpha\mathrm{v} + fu = \qquad\qquad + \alpha\mathrm{v}_{\xi\xi} \qquad (10b)$$

$$\surd(\alpha/\nu)p_\xi \;=\; -\,g\rho\cos\theta \qquad (10c)$$

$$\alpha\rho + u\,\rho_{x\infty} \quad = \qquad\qquad + \alpha\rho_{\xi\xi} \qquad (10d)$$

with boundary conditions $\qquad u, \mathrm{v}_\xi, \rho_\xi \to 0$ as $\xi \to \infty; \qquad u = 0, \mathrm{v} = 0, \rho_\xi = 0$ at $\xi = 0$

(the forcing of ρ_1 by $\rho_{z\infty}$ is not a growing component).

Writing $f\rho - \mathrm{v}\rho_{x\infty} = \psi$, and eliminating u from (10b) and (10d), we have

$$\psi_{\xi\xi} - \psi \quad = \; 0.$$

The growing exponential solution in ξ violates the condition as $\xi \to \infty$, hence

$$f\rho - \mathrm{v}\,\rho_{x\infty} \quad = \quad \psi \;=\; f\rho(0)e^{-\xi} \qquad (11).$$

Writing $\chi = u + \mathrm{v}\surd(b\text{-}1)$, (10a) + (10b)$\surd(b\text{-}1)$ using (11) and the definitions of b, μ gives

$$\chi_{\xi\xi} - \chi - \alpha^{-1}\mu\,\chi \;=\; -g/(\alpha\rho_0)\sin\theta\,\rho(0)\,e^{-\xi} \qquad (12).$$

a. "Critical" b = 1

If $b = 1$, then $u = \chi = g/(2\alpha\rho_0) \sin\theta\, \rho(0)\, \xi\, e^{-\xi}$
from (12), and from (10b)

$$v_{\xi\xi} - v = fg/(2\alpha^2\rho_0) \sin\theta\, \rho(0)\, \xi\, e^{-\xi}, \quad \text{hence } v = -fg/(8\alpha^2\rho_0) \sin\theta\, \rho(0)\, (\xi^2 + \xi)\, e^{-\xi}.$$

For u and v we have again discarded the growing exponential in ξ. Then from (11)

$$\rho = v\, \rho_{x\infty}/f + \rho(0)\, e^{-\xi} = \rho(0)\, (\xi^2 + \xi + 1)\, e^{-\xi}.$$

Then (10a) requires that the growth rate $\alpha = f/\sqrt{8}$; u, v, ρ, and p from (10c), satisfy the boundary conditions at $\xi = 0$, $\xi \to \infty$ and are all determined up to the arbitrary common factor $\rho(0)$.

b. Larger b > 1

If $b > 1$, solutions to the homogeneous form of (12) are $\exp[\pm\xi_+]$ where $\xi_\pm = \xi\sqrt{(1 \pm \mu/\alpha)}$ as before. However, only the negative exponent is acceptable as $\xi \to \infty$. Hence from (12)

$$u + v\sqrt{(b-1)} = \chi = g/(\mu\rho_0) \sin\theta\, \rho(0)\, [\exp(-\xi) - \exp(-\xi_+)] \quad (13)$$

and from (10b)

$$v_{\xi\xi} - v + \alpha^{-1}\mu v = gf/(\alpha\mu\rho_0) \sin\theta\, \rho(0)\, [\exp(-\xi) - \exp(-\xi_+)] \quad (14).$$

The solution of (14) depends on whether or not $\alpha = \mu$.

If $\alpha \neq \mu$, then (using the decaying exponential as $\xi \to \infty$ and v = 0 at $\xi = 0$)

$$v = g/(f\rho_0(b-1)) \sin\theta\, \rho(0)\, [\exp(-\xi) - ½\exp(-\xi_+) - ½\exp(-\xi_-)] \quad (15a).$$

Then u, ρ and p follow from (13), (11) and (10c) in turn:

$$u = g/(\mu\rho_0) \sin\theta\, \rho(0)/2\, [\exp(-\xi_-) - \exp(-\xi_+)] \quad (15b)$$

$$(b-1)\rho = -\rho(0)\, \{e^{-\xi} - b/2\, [\exp(-\xi_-) + \exp(-\xi_+)]\} \quad (15c).$$

The bottom boundary condition $0 = \rho_\xi|_0$ gives $0 = 1 - b/2\, [\sqrt{(1+\mu/\alpha)} + \sqrt{(1-\mu/\alpha)}]$ determining $\alpha = \alpha_p$ as in (B.2). [This analysis is closely parallel to the Laplace Transform analysis, section 3.1].

If $\alpha = \mu$, then (14) gives

$$v = g/(f\rho_0(b-1)) \sin\theta\, \rho(0)\, [\exp(-\xi) - ½\exp(-\xi_+) - ½ + v'\xi] \quad (16a)$$

where v′ has still to be found. Then u, ρ and p follow from (13), (11) and (10c) in turn:

$$u = g/(\mu\rho_0) \sin\theta\, \rho(0)/2\, [1 - \exp(-\xi_+) - 2v'\xi] \quad (16b)$$

$$(b-1)\rho = -\rho(0)\, \{e^{-\xi} - b/2\, [1 + \exp(-\xi_+)] + bv'\xi\} \quad (16c).$$

The bottom boundary condition $0 = \rho_\xi|_0$ gives $0 = 1 - b/\sqrt{2} - bv'$ determining $v' = b^{-1} - 2^{-1/2}$.

If $1 < b < \sqrt{2}$, then $\mu < \alpha_p$ so that the boundary-layer form (15) grows fastest. If $b > \sqrt{2}$, then $\mu > \alpha_p$ so that the form (16) grows fastest, including terms that do not tend to zero outside the bottom boundary layer. If b is very large, then from (16) v becomes small; $\mu \sim \lambda$ and

$$\{\rho, -\rho_{x\infty}u/\lambda\} \sim \{1 + e^{-\xi\sqrt{2}} + \xi\sqrt{2},\ 1 - e^{-\xi\sqrt{2}} + \xi\sqrt{2}\} \quad (17).$$

The form (17) with time-dependence $e^{\mu t}$ matches (C.3), apart from the extra time-dependent factor $t^{-3/2}$ in (C.3) which is the partial result of the initial value problem. As fastest-growing asymptotic forms, the two are consistent: the discrepancy is only through the time-derivatives; differentiating the factor $t^{-3/2}$ gives additional terms in $t^{-5/2}\, e^{\mu t}$ which are relatively small for large times t and were neglected in (C.3).

Thus in each case the form and growth rate of the (fastest) exponentially-growing component is confirmed by this simpler derivation.

5. Discussion

Intuitively, a strong combination of slope and horizontal density gradient (greater density in shallower water) is expected to give dense water acceleration down the slope. In the absence of frictional or diffusive effects, such acceleration by the pressure field is countered by the Coriolis force, enabling a geostrophic balance for along-slope flow. Friction and diffusion tend to release the geostrophic constraint and favour waters of different densities "seeking their own level", e.g. Shapiro and Hill (1997) for this context. Up to the threshold $b \equiv \lambda^2/f^2 = 1$, the release is only to the extent of a steady bottom boundary layer; the boundary layer thickness increases indefinitely as $b \to 1$. Then "modified intuition" suggests that a stronger combination of horizontal density gradient and bottom slope causes down-slope acceleration. The initial value problem now provides the best evidence that such down-slope acceleration does indeed occur, in a bottom boundary layer, for a strong combination of horizontal density gradient and slope ($b \geq 1$). Other components of the solution, not confined to a bottom boundary layer, also accelerate less rapidly down the slope.

The initial-value problem was only treated fully for $b < \sqrt{2}$. As $b \to \sqrt{2}$, the fastest-accelerating boundary layer becomes thick (figure 3).

Whilst the initial-value problem was not treated fully for $\sqrt{2} < b < \infty$, the treatment of possible forms of exponentially increasing flow (section 4) covered all $b \geq 1$. For $b > \sqrt{2}$, the acceleration of the bottom-boundary-layer form was found to be slower than the acceleration [as $f\sqrt{(b\text{-}1)}$] of a form not confined near the bottom boundary. Absence of the Coriolis constraint corresponds to $b \to \infty$; then the intensification rate for the down-slope flow is approximately λ. This agrees with the fastest-growing outcome of the initial-value problem with $f = 0$. Approximately, the threshold ($b = 1$) for acceleration corresponds to $\lambda \sim f$; for $\lambda \sim 2^{1/4}f$ the gravity-induced acceleration throughout the water column overtakes the Coriolis-regulated acceleration in the bottom boundary layer (which in any case has become thick).

An idealized and therefore limited problem has been analysed here. Section 4 is particularly restricted in its simplifying assumption of exponential time-dependence. Whilst this confirms the fastest-growth forms of the more general initial-value analysis, there was no *a priori* reason that they would be found by the simpler section 4 analysis. Indeed, this analysis does not find other growing components of the initial-value solution. The initial-value analysis itself has idealised aspects: the base-state geostrophically-balanced mean flow; the focus on the bottom boundary layer, neglect of the sea surface; the focus on initial instability rather than the form of developed flow that emerges and may depend on various contextual factors. These idealized aspects are now discussed in turn.

The base-state mean flow has a lowest-order geostrophic balance; the only imbalance is in the bottom boundary condition for density. This choice is primarily for simplicity; it emphasises that the resulting instability is an inherent outcome of the combination of horizontal density gradient and bottom slope. Given that the imbalance is only in the frictional and diffusive bottom boundary layer, the adjustment process is subject to important friction and diffusion. This contrasts with geostrophic adjustment or "dam-break" problems where the initial state has geostrophic imbalance through the water column and inertial oscillations result (for example).

The base state with geostrophic shear necessarily has an associated stress $\nu \mathrm{v}_z = -g\nu\rho_H/(\rho_0 f^2)$ which is uniform in z. At the bed, bringing the base-state flow v to zero corresponds to the unforced arrested boundary layer (after evolution) as discussed by Garrett, McCready and Rhines (1993). To bring this stress to zero at a free surface, for a strictly unforced base-state mean flow with no surface stress (forcing), a surface Ekman layer can be invoked.

The analysis focuses on the bottom boundary layer where the assumptions hold most closely (especially, down-slope uniformity, ignoring effects of (i) the sea surface which is not parallel to the sloping bottom and (ii) the necessarily finite lateral extent of slope and density gradient). Solutions have been chosen to be as limited as possible in their behaviour away from the bottom; however, there are contributions not confined near the bottom boundary; their form can only be regarded as suggestive. The slower-growing but spatially unconfined contribution in $1 < b < \sqrt{2}$, and the main growing solution in $b > \sqrt{2}$, really require consideration of the whole water column. Analysis for baroclinic instability should also include the whole water column and consider instability forms that vary (sinusoidally) along the slope. Regarding the finite lateral extent (ii), we argue that transition to a adjacent regions (where base-state density gradients differ) is narrow, $O(\nu_H/f)^{1/2}$ (the lateral equivalent of an Ekman layer). Then the analysis applies to most of the sloping sector (bottom boundary layer) in the upper panel of Figure 1.

We have considered initial acceleration (from a steady state) rather than the developed flow that emerges. The latter is exemplified (without rotation) in Baines (2005), distinguishing detraining gravity currents (on small slopes with small density differences) from entraining plumes (larger slopes and density differences), but with vertical stratification. In realistic oceanic contexts, greater density at depth will eventually inhibit accelerating down-slope flow and divert it to the interior; thus we effectively consider an upper-slope stage in the flow development, above the influence of increased oceanic density with depth, but with rotation as a constraint. In using simply a diffusivity to represent fluid exchange and mixing, we effectively have both entrainment to and detrainment from any denser down-slope flow. Entrainment is implied by the continual down-slope adjustment of density towards the ambient value. Detrainment is implied by the accompanying uniform down-slope transport in any steady state and more generally by the symmetry of the diffusive formulation. A more sophisticated formulation is needed for developed flow.

Developed flow is likely to limit b (to values less than $\sqrt{2}$) in the problem as posed. Down-slope flow of the denser shelf water tends to reduce the up-slope density gradient (in favour of vertical stratification), i.e. dense water flow down the slope tends to reduce the conditions driving it. Assuming continuation of the conditions (e.g. uniform heat loss from shoaling depth) causing the density gradient to increase to $b \geq 1$, an "equilibrium" with b close to 1 seems likely; even if steady down-slope flow in $b < 1$ fails to counteract the increase of density gradient, any increase of b greater than 1 implies ever faster acceleration of down-slope flow tending to reduce the density gradient and hence b back towards 1. In a developed state, the down-slope density-anomaly transport should match the rate of dense water formation; the form of the down-slope flow may also be affected by any entrainment that counters the tendency to accelerate.

In almost all observed cases, $b << 1$, i.e. $|\rho_H|\, g \sin\theta << \rho_0 f^2/\cos\theta$ (Ivanov *et al.* 2004; noting that if the density gradient is strictly horizontal, ρ_H, then $\rho_{x\infty} = \rho_H \cos\theta$). This reflects the fact that observations are typically during preconditioning (cooling and/or salination by brine rejection or evaporation has not built up a sufficient density gradient to initiate cascading) or after any accelerating dense-water flow has reduced the up-slope density gradient. If the means of increasing shallower water density is slow, then continual quasi-steady down-slope flow of denser water in the bottom boundary layer, as described by Shapiro and Hill (1997), may suffice to prevent the up-slope gradient increasing to the critical value for acceleration.

The predicted intensification rate $f/\sqrt{8}$ for the accelerating layer is slow, corresponding to an e-folding time ~ 6 hours at mid-to-high latitudes.

Stronger gradients and hence faster intensification may occur locally in gullies or canyons indenting the slope. Side-walls in a down-slope gully or canyon relax the Coriolis constraint that inhibits down-slope flow. Such topography also tends to juxtapose steep slopes and the up-slope density gradient to give locally-large values of λ (or b) and hence down-slope acceleration.

For example, if relative density differences overall are 10^{-3} and $g \sim 10$ m s^{-2}, then variations over 100 km (corresponding slope $\sin\theta \sim 0.01$) give $\lambda \sim 3 \times 10^{-5}$ s^{-1} $< f$ at mid-latitudes, but variations over 10 km with slope 0.1 give $\lambda \sim 3 \times 10^{-4}$ s^{-1} $> f$.

There remains scope to explore (probably numerically) instability forms with non-zero wave number along the slope, and to carry out numerical and laboratory experiments for comparison.

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APPENDIX A

Evaluation of inverse Laplace transform for b = 1

In the notation of section 3 with $\xi \equiv (\alpha/v)^{1/2}z$ to make all occurrences of α explicit, we have

$$(\rho_1, u_1, \mathrm{v}_1)(z,t) = (2\pi\mathrm{i})^{-1} \int_L (\rho, u, \mathrm{v})\, \mathrm{e}^{\alpha t}\, \mathrm{d}\alpha$$

where

$$\rho = \rho_{z\infty}(\alpha^2 - f^2/8)^{-1}[\sqrt{(v\alpha)} + f^2 z/(8\alpha) + f^2z^2/(8\sqrt{(v\alpha)})]\, \mathrm{e}^{-z\sqrt{(\alpha/v)}},$$

$$u = -\rho_{z\infty}/\rho_{x\infty}(\alpha^2 - f^2/8)^{-1} f^2\, z/2\, \mathrm{e}^{-z\sqrt{(\alpha/v)}},$$

$$\mathrm{v} = +\rho_{z\infty}/\rho_{x\infty}(8\alpha^2 - f^2)^{-1} f^3[z/\alpha + z^2/\sqrt{(v\alpha)}]\, \mathrm{e}^{-z\sqrt{(\alpha/v)}}.$$

The integrand has poles at $\alpha = 0, \pm f/\sqrt{8}$ and a branch point at $\alpha = 0$. The initial integral path is shown in figure 5 as A with $a > f/\sqrt{8}$ for validity. For large (positive or negative) Im(α), the integrand scales as $|\alpha|^{-3/2}$ and the path of integration can be modified. Contributions in deforming the path from A (a-iT to a+iT) to B (over which the integral is arbitrarily small) are from the three poles and the two integral pairs $\int_{C+D}$, $\int_{E+F}$, where the lower path of the pair is in the positive sense and the upper path in the negative sense, both being close to the real axis.

The (ρ, v) terms in z/α give the poles at α=0. Replacing the factor $(\alpha^2 - f^2/8)^{-1}$ by $-8/f^2$, these approximated terms are evaluated as in Carslaw and Jaeger (1947) page 91, resulting in terms

$$({}^{a}\rho_1, {}^{a}\mathrm{v}_1) = -(\rho_{z\infty}, f\rho_{z\infty}/\rho_{x\infty})\, z\ \mathrm{erfc}\,[z/\sqrt{(4vt)}] \tag{A.1}$$

where $\mathrm{erfc}\,[z/\sqrt{(4vt)}] \equiv 1 - (\pi vt)^{-1/2}\int_0^z \exp[-z'^2/(4vt)]\,\mathrm{d}z' \equiv 1 - \mathrm{erf}\,[z/\sqrt{(4vt)}]$;

erf is the error function.

The remainder of the solution is $\{\rho_{z\infty}^{-1}\, {}^{r}\rho_1, \rho_{x\infty}/\rho_{z\infty}({}^{r}u_1, {}^{r}\mathrm{v}_1)\}$

$$= (2\pi\mathrm{i})^{-1}\int_L \mathrm{e}^{\alpha t - z\sqrt{(\alpha/v)}}(\alpha^2 - f^2/8)^{-1}\{\sqrt{(v\alpha)} + z\alpha + f^2z^2/(8\sqrt{(v\alpha)}),\ -f^2z/2,\ fz\alpha + z^2f^3/(8\sqrt{(v\alpha)})\}\,\mathrm{d}\alpha$$

The poles at $\alpha = f/\sqrt{8}$ result in growing terms

$$\{f\rho_{z\infty}^{-1}\, {}^{g}\rho_1, \rho_{x\infty}/\rho_{z\infty}({}^{g}u_1, {}^{g}\mathrm{v}_1)\} = \sqrt{(2^{1/2}vf)}\,\{1 + \xi + \xi^2, -2^{1/2}\xi, \xi + \xi^2\}\mathrm{e}^{ft/\sqrt{8} - \xi} \tag{A.2}$$

where $\xi = (8^{-1/2}f/v)^{1/2}z$ [$= (\alpha/v)^{1/2}z$ as before].

The poles at $\alpha = -f/\sqrt{8}$ and integrals on C, D result in terms decaying at least as fast as $\mathrm{e}^{-ft/\sqrt{8}}$.

The integrals on E, F are $\{\rho_{z\infty}^{-1}\, {}^{r}\rho_1, f^{-1}\rho_{x\infty}/\rho_{z\infty}({}^{r}u_1, {}^{r}\mathrm{v}_1)\}$

$$= (2\pi\mathrm{i})^{-1}\int_0^{f/\sqrt{8}}(s^2 - f^2/8)^{-1}\,\mathrm{e}^{-st + z\mathrm{i}\sqrt{(s/v)}}\{-\mathrm{i}\sqrt{(vs)} - zs + \mathrm{i}f^2z^2/(8\sqrt{(vs)}),\ -fz/2,\ -zs + \mathrm{i}z^2f^2/(8\sqrt{(vs)})\}\,\mathrm{d}s$$

+ complex conjugate (c.c.).

Changing variable to $x = \sqrt{(st)}$, $t\mathrm{d}s = 2x\mathrm{d}x$,

$$\{\rho_{z\infty}^{-1}\, {}^{r}\rho_1, f^{-1}\rho_{x\infty}/\rho_{z\infty}({}^{r}u_1, {}^{r}\mathrm{v}_1)\}$$

$$= (\pi\mathrm{i})^{-1}\int_0^{\sqrt{(ft/\sqrt{8})}}(x^4t^{-2} - f^2/8)^{-1}\exp[-(x - \mathrm{i}z/\sqrt{(4vt)})^2 - z^2/(4vt)]$$

$$\times\{-\mathrm{i}x^2\sqrt{(vt^{-3})} - zx^3/t^2 + \mathrm{i}f^2z^2/(8\sqrt{(vt)}),\ -fzx/2t,\ -zx^3/t^2 + \mathrm{i}z^2f^2/(8\sqrt{(vt)})\}\mathrm{d}x \quad + \text{c.c.}$$

$$\sim 8\mathrm{i}f^{-2}/\pi\int_0^{\infty}\exp[-(x - \mathrm{i}z/\sqrt{(4vt)})^2 - z^2/(4vt)]$$

$$\times\{-\mathrm{i}x^2\sqrt{(vt^{-3})} - x^2z^2\mathrm{i}/\sqrt{(4vt^5)} - zx/t^2 + \mathrm{i}f^2z^2/(8\sqrt{(vt)}),\ -fzx/2t,\ -zx/t^2 - \mathrm{i}z^2x^2/\sqrt{(4vt^5)} + \mathrm{i}z^2f^2/(8\sqrt{(vt)})\}\mathrm{d}x$$

+ c.c. (for large t)

$= 8\mathrm{i}f^2/\pi\int^{\infty}_{-\mathrm{i}z/\sqrt{(4\nu t)}} \exp[-y^2-z^2/(4\nu t)]$

$\times\{-\mathrm{i}[y^2+y\mathrm{i}z/\sqrt{(\nu t)}-z^2/(4\nu t)]\sqrt{(\nu t^{-3})}(1+z^2/(2\nu t))-zy/t^2-\mathrm{i}z^2/\sqrt{(4\nu t^5)}+\mathrm{i}f^2z^2/(8\sqrt{(\nu t)}),$

$-fzy/2t - \mathrm{i}fz^2/\sqrt{(16\nu t^3)},$

$z[y+\mathrm{i}z/\sqrt{(4\nu t)}]/t^2 - \mathrm{i}z^2[y^2+y\mathrm{i}z/\sqrt{(\nu t)}-z^2/(4\nu t)]/\sqrt{(4\nu t^5)} + \mathrm{i}z^2f^2/(8\sqrt{(\nu t)})\}\, \mathrm{d}y$ + c.c.

We use $\int^{\infty}_{-\mathrm{i}z/\sqrt{(4\nu t)}} y^2\exp(-y^2)\, \mathrm{d}y = [-y/2 \exp(-y^2)]^{\infty}_{-\mathrm{i}z/\sqrt{(4\nu t)}} + [\int^{0}_{-\mathrm{i}z/\sqrt{(4\nu t)}} + \int_0^{\infty}] \exp(-y^2)\, \mathrm{d}y/2,$

$\int^{\infty}_{-\mathrm{i}z/\sqrt{(4\nu t)}} y \exp(-y^2)\, \mathrm{d}y = [-1/2 \exp(-y^2)]^{\infty}_{-\mathrm{i}z/\sqrt{(4\nu t)}}$

$\int_0^{\infty} \exp(-y^2)\, \mathrm{d}y = \sqrt{\pi}/2,$

write $\int^{0}_{-\mathrm{i}z/\sqrt{(4\nu t)}} \exp(-y^2)\mathrm{d}y = \mathrm{erf}[\mathrm{i}z/\sqrt{(4\nu t)}]$ (non-standard notation),

and note that $\mathrm{erf}[\mathrm{i}z/\sqrt{(4\nu t)}]$ is imaginary and imaginary terms cancel their complex conjugates.

Then for large t, $\{\rho_{z\infty}^{-1}\, {}^{\mathrm{r}}\rho_1,\ f^{-1}\rho_{x\infty}/\rho_{z\infty}({}^{\mathrm{r}}u_1,\ {}^{\mathrm{r}}\mathrm{v}_1)\}$

$= 4f^2/\sqrt{\pi} \exp[-z^2/(4\nu t)] \times\{[1/2-z^2/(4\nu t)]\sqrt{(\nu t^{-3})}(1+z^2/(2\nu t))+z^2/\sqrt{(4\nu t^5)}-f^2z^2/(8\sqrt{(\nu t)}),$

$fz^2/\sqrt{(16\nu t^3)},\ z^2/\sqrt{(4\nu t)}/t^2+z^2[1/2-z^2/(4\nu t)]/\sqrt{(4\nu t^5)}-z^2f^2/(8\sqrt{(\nu t)})\}$

$= 4f^2 (\nu t^{-3}/\pi)^{1/2} \exp[-z^2/(4\nu t)]$

$$\times\{1/2 + (1-f^2t^2/4)z^2/(2\nu t) - z^4/(8\nu^2t^2),\ ftz^2/(4\nu t),\ z^2/(4\nu t)\, (3-f^2t^2/2) - z^4/(8\nu^2t^2)\} \qquad \text{(A.3).}$$

The total solution for large time is the sum of terms decaying at least as fast as $\mathrm{e}^{-ft/\sqrt{8}}$ plus the combination of $({}^{\mathrm{a}}\rho_1, {}^{\mathrm{a}}\mathrm{v}_1)$, $\{{}^{\mathrm{g}}\rho_1, {}^{\mathrm{g}}u_1, {}^{\mathrm{g}}\mathrm{v}_1\}$ and $\{{}^{\mathrm{r}}\rho_1, {}^{\mathrm{r}}u_1, {}^{\mathrm{r}}\mathrm{v}_1\}$ given by (A.1), (A.2) and (A.3) respectively. In fact all terms of (A.3) decay at least as fast as $t^{-1/2}$ at large times so that the only terms remaining at large times are $({}^{\mathrm{a}}\rho_1, {}^{\mathrm{a}}\mathrm{v}_1)$ as in (A.1) and $\{{}^{\mathrm{g}}\rho_1, {}^{\mathrm{g}}u_1, {}^{\mathrm{g}}\mathrm{v}_1\}$ as in (A.2).

APPENDIX B

Evaluation of inverse Laplace transform for $b > 1$

In the notation of section 3 we have

$$(\rho_1, u_1, \mathrm{v}_1)(z,t) = (2\pi\mathrm{i})^{-1} \int_{\mathrm{L}} (\rho, u, \mathrm{v})\, \mathrm{e}^{\alpha t}\, \mathrm{d}\alpha$$

where from (6) we have [with $\xi_{\pm} \equiv q_{\pm}\xi$, $A(\alpha) = \{2 - b[(1+\mu/\alpha)^{1/2} + (1-\mu/\alpha)^{1/2}]\}$, $\mu = f(b-1)^{1/2}$]

$$\begin{Bmatrix} \rho \\ f\rho_0(g\sin\theta)^{-1} u \\ f\rho_0(g\sin\theta)^{-1}\mathrm{v} \end{Bmatrix} = \rho_{z\infty}\sqrt{(\nu/\alpha^3)}\, [A(\alpha)]^{-1} \times \begin{Bmatrix} 2\exp(-\xi) - b[\exp(-\xi_+)+\exp(-\xi_-)], \\ \mu f^{-1} [\exp(-\xi_+)-\exp(-\xi_-)], \\ \exp(-\xi_+) + \exp(-\xi_-) - 2\exp(-\xi) \end{Bmatrix}.$$

For large (positive or negative) $\mathrm{Im}(\alpha)$, the integrand scales as $|\alpha|^{-3/2}$ and the path of integration can be modified as in figure 6.

There is an obvious pole at $\alpha = 0$ and branch points at $\alpha = 0, \pm\mu$. There may also be any poles found by rearranging $A(\alpha)=0$ and squaring twice, which can introduce spurious zeros: at most $\alpha = \pm f b^2 (1+b)^{-1/2}/2 \equiv \pm\alpha_{\mathrm{p}}$. Indeed, $\alpha = -\alpha_{\mathrm{p}}$ is not valid because $A(\alpha)$ is complex and non-zero for $\alpha < \mu$. Since (i) $\mu \le \alpha_{\mathrm{p}}$, (ii) $A(\mu) = 2 - b\sqrt{2} > 0$ if and only if $b < \sqrt{2}$ and (iii) $A(\alpha) \to 2-2b < 0$ as $\alpha \to \infty$, then $\alpha = \alpha_{\mathrm{p}}$ is only a pole if $b < \sqrt{2}$. [As b tends to 1, α_{p} tends to $f/\sqrt{8}$ as for b=1].

The contributions in deforming the path from A (a-iT to a+iT) to B (over which the integral is arbitrarily small) are from the three poles and the five integral pairs $\int_{\mathrm{C+D}}$, $\int_{\mathrm{E+F}}$, $\int_{\mathrm{G+H}}$, $\int_{\mathrm{J+K}}$, $\int_{\mathrm{L+M}}$. Here the lower path of the pair is in the positive sense and the upper path in the negative sense, both being close to the real axis.

Near $\alpha = 0$, let $\alpha = \mu r e^{i\varphi}$ ($-\pi<\varphi<\pi$; $r<<1$), $\eta \equiv z\,(\mu/v)^{1/2}$. The integrand for

$$\rho_{z\infty}^{-1}\,\{\rho_1, f\rho_0(g\sin\theta)^{-1}(u_1,v_1)\}$$

is

$$\surd(v/\mu)\,(2\pi b)^{-1}(-1+i)^{-1}\{2 - b(e^{-\eta} + e^{i\eta}),\ \surd(b\text{-}1)(e^{-\eta} - e^{i\eta}),\ e^{-\eta} + e^{i\eta} - 2\}\exp(\mu t r e^{i\varphi})d\varphi \quad (-\pi<\varphi<0)$$
$$+ \surd(v/\mu)\,(2\pi b)^{-1}(-1-i)^{-1}\{2 - b(e^{-\eta}+e^{-i\eta}),\ \surd(b\text{-}1)(e^{-\eta}-e^{-i\eta}),\ e^{-\eta}+e^{-i\eta} - 2\}\exp(\mu t r e^{i\varphi})d\varphi \quad (0<\varphi<\pi)$$
$$+ O(r^{1/2}),$$

i.e. the integral is

$$-\surd(v/\mu)\,b^{-1}\ \{1 - b(e^{-\eta}+\cos\eta\text{-}\sin\eta)/2,\ \surd(b\text{-}1)(e^{-\eta}\text{-}\cos\eta+\sin\eta)/2,\ (e^{-\eta}+\cos\eta\text{-}\sin\eta)/2 - 1\} \quad \text{(B.1)}$$

as $r \to 0$. For small b-1 and η this reduces further to

$$\rho_{z\infty}^{-1}\,\{\rho_1, f\rho_0(g\sin\theta)^{-1}\,(u_1, v_1)\} \sim z\ \{-1, z\surd((b\text{-}1)\mu/v), 1\}.$$

Near the pole $\alpha = \alpha_p$, where $\{A(\alpha)\} = 0$, we have $(\alpha - \alpha_p)\ \{A(\alpha)\}^{-1} \sim [\partial_\alpha\{A(\alpha)\}]^{-1}$ so that the component of $\rho_{z\infty}^{-1}\,\{\rho_1,\ f\rho_0(g\sin\theta)^{-1}\,(u_1, v_1)\}$ growing as $\exp(\alpha_p t)$ is

$$-\,2\surd(v/(b\text{-}1))\ (bf)^{-1}\ (\alpha_p^{\,2}\text{-}\mu^2)^{1/2}\ [(\alpha_p+\mu)^{1/2} - (\alpha_p-\mu)^{1/2}]^{-1} \times\ \exp(\alpha_p t)\ \times$$
$$\{2\exp(-\xi) - b[\exp(-\xi_+)+\exp(-\xi_-)],\ \ \surd(b\text{-}1)[\exp(-\xi_+)\text{-}\exp(-\xi_-)],\ \ \exp(-\xi_+)+\exp(-\xi_-)\text{-}2\exp(-\xi)\} \quad \text{(B.2)}$$

where ξ, $\xi_\pm$ are as before but evaluated at $\alpha = \alpha_p$. For small b-1, (B.2) reduces to

$$(\alpha - \alpha_p)\,\rho_{z\infty}^{-1}\,\{\rho, f\rho_0(g\sin\theta)^{-1}\,(u, v)\}\ \sim\ \surd(2^{1/2}\,v/f)\ \exp(ft/\surd 8)\ \exp(-\xi)\ \{1+\xi+\xi^2,\ \xi\surd 2,\ -\xi - \xi^2\}$$

in agreement with (A.2).

The solution corresponding to $\alpha = -\alpha_p$ decays in time t and with distance ξ from the bottom.

For the integral pairs, the integrand for

$$\left\{\begin{array}{l}\rho_1,\\ f\rho_0(g\sin\theta)^{-1}u_1,\\ f\rho_0(g\sin\theta)^{-1}v_1\end{array}\right\} \text{ is } \frac{\rho_{z\infty}\surd v\,(2\pi i\alpha)^{-1}\,e^{\alpha t}}{[2\surd\alpha - b\surd(\alpha+\mu) - b\surd(\alpha\text{-}\mu)]} \times \left\{\begin{array}{l}2e^{-z\surd(\alpha/v)} - be^{-\surd(\alpha+\mu)z/\surd v} - be^{-\surd(\alpha-\mu)z/\surd v},\\ \surd(b\text{-}1)\,[e^{-\surd(\alpha+\mu)z/\surd v} - e^{-\surd(\alpha-\mu)z/\surd v}],\\ e^{-\surd(\alpha+\mu)z/\surd v} + e^{-\surd(\alpha-\mu)z/\surd v} - 2e^{-z\surd(\alpha/v)}\end{array}\right\}.$$

For the pair L, M, there are no branch cuts, the integrand has the same values immediately above and below the real axis, but the path M reverses L. Hence this integral pair contributes zero.

Next consider GHJK combined. For the pair JK, $0 < \beta \equiv \alpha/\mu < 1$, so $\surd(\alpha\text{-}\mu) = -i[(1\text{-}\beta)\mu]^{1/2}$ on J and $\surd(\alpha\text{-}\mu) = i[(1\text{-}\beta)\mu]^{1/2}$ on K. For the pair GH, $0 < \beta \equiv -\alpha/\mu < 1$, so $\surd(\alpha\text{-}\mu) = -i[(1+\beta)\mu]^{1/2}$ on G and $\surd(\alpha\text{-}\mu) = i[(1+\beta)\mu]^{1/2}$ on H. Also $\surd\alpha = -i[\beta\mu]^{1/2}$ on G, $\surd\alpha = i[\beta\mu]^{1/2}$ on H, $\surd(\alpha+\mu) = [(1\text{-}\beta)\mu]^{1/2}$ on G and H. Then writing $\eta \equiv z\,(\mu/v)^{1/2}$, $\eta\pm \equiv \eta(1\pm\beta)^{1/2}$, the combined integrand over $0 < \beta < 1$ for $2\pi i\,\rho_{z\infty}^{-1}\,\{\rho_1, f\rho_0(g\sin\theta)^{-1}\,(u_1, v_1)\}$ is

(on J) $$\surd v\,\beta^{-1}\,[\,2\surd(\mu\beta) - b\surd(\mu+\mu\beta) + ib\surd(\mu\text{-}\mu\beta)]^{-1}$$
$$\times\ \{2e^{-\eta\surd\beta} - be^{-\eta+} - be^{i\eta-},\ \ \surd(b\text{-}1)\,[e^{-\eta+} - e^{i\eta-}],\ e^{-\eta+} + e^{i\eta-} - 2e^{-\eta\surd\beta}\}e^{\mu\beta t}\,d\beta$$

(on K) $$-\,\surd v\,\beta^{-1}\,[\,2\surd(\mu\beta) - b\surd(\mu+\mu\beta) - ib\surd(\mu\text{-}\mu\beta)]^{-1}$$
$$\times\ \{2e^{-\eta\surd\beta} - be^{-\eta+} - be^{-i\eta-},\ \surd(b\text{-}1)\,[e^{-\eta+} - e^{-i\eta-}],\ e^{-\eta+} + e^{-i\eta-} - 2e^{-\eta\surd\beta}\}e^{\mu\beta t}\,d\beta$$

(on G) $$-\,\surd v\,\beta^{-1}\,[-2i\surd(\mu\beta) + ib\surd(\mu+\mu\beta) - b\surd(\mu\text{-}\mu\beta)]^{-1}$$

$\times \{2e^{i\eta\sqrt{\beta}} - be^{i\eta+} - be^{-\eta-}, \sqrt{(b-1)}\,[e^{-\eta-} - e^{i\eta+}],\ e^{-\eta-} + e^{i\eta+} - 2e^{i\eta\sqrt{\beta}}\}e^{-\mu\beta t}\,d\beta$

(on H) $+ \sqrt{v}\,\beta^{-1}\,[\,2i\sqrt{(\mu\beta)} - ib\sqrt{(\mu+\mu\beta)} - b\sqrt{(\mu-\mu\beta)}]^{-1}$

$\times \{2e^{-i\eta\sqrt{\beta}} - be^{-i\eta+} - be^{-\eta-}, \sqrt{(b-1)}\,[e^{-\eta-} - e^{-i\eta+}], e^{-\eta-} + e^{-i\eta+} - 2e^{-i\eta\sqrt{\beta}}\}e^{\mu\beta t}\,d\beta$

$= \sqrt{v}\,(2\mu\beta)^{-1}\,d\beta\,[2\beta - 2b\sqrt{(\beta+\beta^2)} + b^2]^{-1}$

$\times$ [$(2\sqrt{(\mu\beta)} - b\sqrt{(\mu+\mu\beta)})\,e^{\mu\beta t}\,\{-b, -\sqrt{(b-1)}, 1\}2i\sin\eta-$

$-2ib\sqrt{(\mu-\mu\beta)}\,e^{\mu\beta t}\,\{2e^{-\eta\sqrt{\beta}} - be^{-\eta+} - b\cos\eta-, \sqrt{(b-1)}\,(e^{-\eta+} - \cos\eta-), e^{-\eta+} - 2e^{-\eta\sqrt{\beta}} + \cos\eta-\}$

$-2i(2\sqrt{(\mu\beta)} - b\sqrt{(\mu+\mu\beta)})\,e^{-\mu\beta t}\,\{2\cos\eta\sqrt{\beta} - be^{-\eta-} - b\cos\eta+, \sqrt{(b-1)}\,(e^{-\eta-} - \cos\eta+),$

$e^{-\eta-} - 2\cos\eta\sqrt{\beta} + \cos\eta+\}$

$+2ib\sqrt{(\mu-\mu\beta)}\,e^{-\mu\beta t}\,\{2\sin\eta\sqrt{\beta} - b\sin\eta+, -\sqrt{(b-1)}\sin\eta+, -2\sin\eta\sqrt{\beta} + \sin\eta+\}$];

i.e. the combined integrand for $\rho_{z\infty}^{-1}\,\{\rho_1, f\rho_0(g\sin\theta)^{-1}\,(u_1, v_1)\}$ is

$\sqrt{(\upsilon/\mu)}\,(2\pi\beta)^{-1}\,d\beta\,[2\beta - 2b\sqrt{(\beta+\beta^2)} + b^2]^{-1}$

$\times [(2\sqrt{\beta} - b\sqrt{(1+\beta)})\,e^{\mu\beta t}\,\{-b, -\sqrt{(b-1)}, 1\}\sin\eta-$

$-\,b\sqrt{(1-\beta)}\,e^{\mu\beta t}\,\{2e^{-\eta\sqrt{\beta}} - be^{-\eta+} - b\cos\eta-, \sqrt{(b-1)}\,(e^{-\eta+} - \cos\eta-), e^{-\eta+} - 2e^{-\eta\sqrt{\beta}} + \cos\eta-\}$

$-\,(2\sqrt{\beta} - b\sqrt{(1+\beta)})\,e^{-\mu\beta t}\,\{2\cos\eta\sqrt{\beta} - be^{-\eta-} - b\cos\eta+, \sqrt{(b-1)}\,(e^{-\eta-} - \cos\eta+),\ e^{-\eta-} - 2\cos\eta\sqrt{\beta} + \cos\eta+\}$

$+\,b\sqrt{(1-\beta)}\,e^{-\mu\beta t}\,\{2\sin\eta\sqrt{\beta} - b\sin\eta+, -\sqrt{(b-1)}\sin\eta+, -2\sin\eta\sqrt{\beta} + \sin\eta+\}$].

If b-1 is small and z is not large, then η is $O(\mu^{1/2})$; this combined GHJK integrand in $0<\beta<1$ is approximately

$\sqrt{(v/\mu)}\,(2\pi\beta)^{-1}\,d\beta\,[1 - 2\sqrt{(\beta+\beta^2)} + 2\beta]^{-1}$

$\times$ [$(2\sqrt{\beta} - \sqrt{(1+\beta)})\,e^{\mu\beta t}\,\{-1, -\mu/f, 1\}\sqrt{(1-\beta)}\,\eta$

$+ \sqrt{(1-\beta)}\,e^{\mu\beta t}\,\{\eta(2\sqrt{\beta} - \sqrt{(1+\beta)}), \eta\mu/f\,\sqrt{(1+\beta)}, -\eta(2\sqrt{\beta} - \sqrt{(1+\beta)})\}$

$+ (2\sqrt{\beta} - \sqrt{(1+\beta)})\,e^{-\mu\beta t}\,\{-\eta\sqrt{(1-\beta)}, \eta\mu/f\,\sqrt{(1-\beta)}, \eta\sqrt{(1-\beta)}\}$

$+ \sqrt{(1-\beta)}\,e^{-\mu\beta t}\,\{\eta(2\sqrt{\beta} - \sqrt{(1+\beta)}), -\eta\mu/f\,\sqrt{(1+\beta)}, -\eta(2\sqrt{\beta} - \sqrt{(1+\beta)})\}$] $+ O(\eta^2\mu/f, \eta^3)$

$= 2\eta\,\sqrt{(v\mu)}\,(\pi\beta f)^{-1}\,d\beta\,[1-2\sqrt{(\beta+\beta^2)}+2\beta]^{-1}\,\sqrt{(1-\beta)}\,\{0, [\sqrt{(1+\beta)} - \sqrt{\beta}], 0\}\,\sinh(\mu\beta t) + O(\eta^2\mu/f, \eta^3)$

which is integrable at any given time, and is small as b-1 → 0.

If β (> 0) is small (and $\eta\sqrt{\beta} \ll 1$) then the combined GHJK integrand is approximately

$\sqrt{(v/\mu)}\,(2\pi\beta b)^{-1}\,d\beta$

$\times$ [$-\,e^{\mu\beta t}\,\{-b, -\sqrt{(b-1)}, 1\}\sin\eta$

$-\,e^{\mu\beta t}\,\{2e^{-\eta\sqrt{\beta}} - be^{-\eta} - b\cos\eta, \sqrt{(b-1)}\,(e^{-\eta} - \cos\eta), e^{-\eta} - 2e^{-\eta\sqrt{\beta}} + \cos\eta\}$

$+\,e^{-\mu\beta t}\,\{2\cos\eta\sqrt{\beta} - be^{-\eta} - b\cos\eta, \sqrt{(b-1)}\,(e^{-\eta} - \cos\eta),\ e^{-\eta} - 2\cos\eta\sqrt{\beta} + \cos\eta\}$

$+\,e^{-\mu\beta t}\,\{2\sin\eta\sqrt{\beta} - b\sin\eta, -\sqrt{(b-1)}\sin\eta, -2\sin\eta\sqrt{\beta} + \sin\eta\}$]

+ integrable terms $O(\beta^{-1/2})$

$$= \quad \surd(\nu/\mu)\,(2\pi\beta b)^{-1}\,d\beta$$
$$\times \{e^{\mu\beta t}[b(\sin\eta + \cos\eta + e^{-\eta}) - 2e^{-\eta\surd\beta}] - e^{-\mu\beta t}[b(\sin\eta + \cos\eta + e^{-\eta}) - 2\cos\eta\surd\beta - 2\sin\eta\surd\beta)],$$
$$(e^{\mu\beta t} - e^{-\mu\beta t})\surd(b\text{-}1)\,(\sin\eta + \cos\eta - e^{-\eta}),$$
$$- e^{\mu\beta t}[\sin\eta + \cos\eta + e^{-\eta} - 2e^{-\eta\surd\beta}] + e^{-\mu\beta t}[\sin\eta + \cos\eta + e^{-\eta} - 2\cos\eta\surd\beta - 2\sin\eta\surd\beta)]\} + O(\beta^{-1/2})$$
$$= \quad \surd(\nu/\mu)\,(\pi\beta b)^{-1}\,\sinh(\mu\beta t)\,d\beta$$
$$\times \{b(\sin\eta+\cos\eta+e^{-\eta}) - 2,\ (\sin\eta+\cos\eta-e^{-\eta})\,\mu/f,\ 2 - (\sin\eta+\cos\eta+e^{-\eta})\} \qquad + O(\eta\beta^{-1/2}\cosh(\mu\beta t))$$

showing that the singularities on the separate paths (G, J, K, H) as $\beta \rightarrow 0$ cancel; the combined integrand is integrable; there is no discrete contribution from the apparent pole $\beta = 0$.

For β near 1, let $\beta = 1\text{-}\delta$; the integrand for $\rho_{z\infty}^{-1}\{\rho_1, f\rho_0(g\sin\theta)^{-1}(u_1, v_1)\}$ is approximately

$$\surd(\nu/\mu)\,(2\pi)^{-1}\,d\beta\,[(b\text{-}\surd2)^2 + \delta(3b/\surd2 - 2)]^{-1}$$
$$\times\ [\ (2\text{-}b\surd2 - \delta(1\text{-}2^{-3/2}b))\,e^{\mu\beta t}\,\{\text{-}b,\ \text{-}\mu/f,\ 1\}\sin\eta\text{-}$$
$$- b\surd\delta\,e^{\mu\beta t}\,\{2e^{-\eta} - be^{-\eta\surd2} - b\cos\eta\surd\delta,\ [e^{-\eta\surd2} - \cos\eta\text{-}]\,\mu/f,\ e^{-\eta\surd2} - 2e^{-\eta} + \cos\eta\surd\delta\}$$
$$- (2\text{-}b\surd2\text{-}\delta(1\text{-}2^{-3/2}b))\,e^{-\mu\beta t}\,\{2\cos\eta\text{-}be^{-\eta\text{-}}\text{-}b\cos\eta\surd2,\ [e^{-\eta\text{-}}\text{-}\cos\eta\surd2]\,\mu/f,\ e^{-\eta\text{-}}\text{-}2\cos\eta\text{+}\cos\eta\surd2\}$$
$$+ b\surd\delta\,e^{-\mu\beta t}\,\{2\sin\eta - b\sin\eta\surd2,\ -(\mu/f)\sin\eta\surd2,\ -2\sin\eta + \sin\eta\surd2\} \qquad].$$

The factors $(2 - b\surd2 - \delta(1\text{-}2^{-3/2}b))$ or $\surd\delta$ ensure integrability, even when $b{=}\surd2$, despite possibly large values of $[(b\text{-}\surd2)^2 + \delta(3b/\surd2 - 2)]^{-1}$ as $\delta \rightarrow 0$ ($\beta \rightarrow 1$) in the integration. If $b \neq \surd2$, the fastest-growing part, determined by the factor $e^{\mu(1\text{-}\delta)t}$, is approximately

$$\surd(\nu/\mu)\,(2\pi)^{-1}\,e^{\mu t}\,(b\text{-}\surd2)^{-2}\int_0^{\Delta} e^{-\mu\delta t}\,[\{\text{-}b,\ \text{-}\mu/f,\ 1\}(2\text{-}b\surd2)\,\sin\eta\,\delta^{1/2}$$
$$- b\surd\delta\{2e^{-\eta} - be^{-\eta\surd2} - b,\ (e^{-\eta\surd2}\text{-}1)\mu/f,\ e^{-\eta\surd2} - 2e^{-\eta} + 1\}]\,d\delta.$$

We use $\int_0^{\Delta} \delta^{1/2}\,e^{-\mu\delta t}\,d\delta = (\mu t)^{-3/2}\int_0^{Y} 2y^2\exp(-y^2)\,dy = -(\mu t)^{-1}\Delta^{1/2}\exp[-\Delta\mu t] + (\mu t)^{-3/2}\pi^{1/2}/2\ \mathrm{erf}(Y)$

$\sim\ (\mu t)^{-3/2}\pi^{1/2}/2$ as $t \rightarrow \infty$, any fixed $\Delta < 1$, [here $Y = (\Delta\mu t)^{1/2}$]

and $\int_0^{\Delta} \delta^{n+1/2}\,e^{-\mu\delta t}\,d\delta = \int_0^{d} \delta^{n+1/2}\,e^{-\mu\delta t}\,d\delta + \int_d^{\Delta} \delta^{n+1/2}\,e^{-\mu\delta t}\,d\delta < d^{n+3/2} + \Delta^{n+3/2}e^{-d\mu t}$.

Hence for large t and $n \geq 1$ we can choose $d = t^{-\alpha}$ ($0.6 < \alpha < 1$) so that as $t \rightarrow \infty$, any fixed $\Delta < 1$,

$$\int_0^{\Delta} \delta^{n+1/2}\,e^{-\mu\delta t}\,d\beta < d^{n+3/2} + \Delta^{n+3/2}e^{-d\mu t} \sim d^{n+3/2} << (\mu t)^{-3/2}\pi^{1/2}/2 \sim \int_0^{\Delta} \delta^{1/2}\,e^{-\mu\delta t}\,d\beta.$$

Thus the fastest-growing part of $\{\rho_1, f\rho_0(g\sin\theta)^{-1}(u_1, v_1)\}$ is approximately

$$\rho_{z\infty}\surd[\nu/(2^4\mu^4t^3\pi)]\,e^{\mu t}\,(b\text{-}\surd2)^{-2}\ \times$$
$$[\{\text{-}b,\ \text{-}\mu/f,\ 1\}(2\text{-}b\surd2)\eta - b\{2e^{-\eta}\text{-}be^{-\eta\surd2}\text{-}b,\ (e^{-\eta\surd2}\text{-}1)\mu/f,\ e^{-\eta\surd2}\text{-}2e^{-\eta}\text{+}1\}] \qquad (B.3).$$

The pair JK contributes the terms in $e^{\mu\beta t}$. The growth rate is slow for small $\mu = f\surd(b\text{-}1)$; much slower than the growth rate from the pole at $\alpha = \alpha_p$. As b increases from 1 to $\surd2$, the growth rate of (B.3) increases from 0 to $f\surd(\surd2\text{-}1) = \alpha_p$ (at $b{=}\surd2$), the growth rate from the pole.

For the pairs CD and EF, $\surd(\alpha+\mu) = \text{-}i[(\beta\text{-}1)\mu]^{1/2}$ on C, E and $\surd(\alpha+\mu) = i[(\beta\text{-}1)\mu]^{1/2}$ on D, F, where $\beta > \alpha_p/\mu$ on C, $1 < \beta \equiv \text{-}\alpha/\mu < \alpha_p/\mu$ on E. Also $\surd\alpha = \text{-}i[\beta\mu]^{1/2}$ on C, E and $\surd\alpha = i[\beta\mu]^{1/2}$ on D, F; $\surd(\alpha\text{-}\mu) = \text{-}i[(\beta+1)\mu]^{1/2}$ on C, E and $\surd(\alpha\text{-}\mu) = i[(\beta+1)\mu]^{1/2}$ on D, F. Writing

$$\eta \equiv z\,(\mu\beta/\nu)^{1/2}, \qquad \eta_{\pm} \equiv z\,[\mu(\beta\pm1)/\nu]^{1/2},$$

the integrand for $2\pi i\,\rho_{z\infty}^{-1}\,\{\rho_1, f\rho_0(g\sin\theta)^{-1}\,(u_1, v_1)\}$ is (on E: $1 < \beta < \alpha_p/\mu$) $\quad - \mathrm{i}\,\surd(\nu/\mu)\,\beta^{-1}\,\mathrm{d}\beta\,\exp(-\beta t\mu)\,[2\surd\beta - b\surd(\beta+1) - b\surd(\beta-1)]^{-1}$

$$\times\ \{2e^{i\eta} - be^{i\eta-} - be^{i\eta+},\ \surd(b\text{-}1)\,[e^{i\eta-} - e^{i\eta+}],\ e^{i\eta-} + e^{i\eta+} - 2e^{i\eta}\}$$

(on F: $1 < \beta < \alpha_p/\mu$) $\quad - \mathrm{i}\,\surd(\nu/\mu)\,\beta^{-1}\,\mathrm{d}\beta\,\exp(-\beta t\mu)\,[2\surd\beta - b\surd(\beta+1) - b\surd(\beta-1)]^{-1}$

$$\times\ \{2e^{-i\eta} - be^{-i\eta-} - be^{-i\eta+},\ \surd(b\text{-}1)\,[e^{-i\eta-} - e^{-i\eta+}],\ e^{-i\eta-} + e^{-i\eta+} - 2e^{-i\eta}\}$$

(on C: $\alpha_p/\mu < \beta$) $\quad - \mathrm{i}\,\surd(\nu/\mu)\,\beta^{-1}\,\mathrm{d}\beta\,\exp(-\beta t\mu)\,[2\surd\beta - b\surd(\beta+1) - b\surd(\beta-1)]^{-1}$

$$\times\ \{2e^{i\eta} - be^{i\eta-} - be^{i\eta+},\ \surd(b\text{-}1)\,[e^{i\eta-} - e^{i\eta+}],\ e^{i\eta-} + e^{i\eta+} - 2e^{i\eta}\}$$

(on D: $\alpha_p/\mu < \beta$) $\quad - \mathrm{i}\,\surd(\nu/\mu)\,\beta^{-1}\,\mathrm{d}\beta\,\exp(-\beta t\mu)\,[2\surd\beta - b\surd(\beta+1) - b\surd(\beta-1)]^{-1}$

$$\times\ \{2e^{-i\eta} - be^{-i\eta-} - be^{-i\eta+},\ \surd(b\text{-}1)\,[e^{-i\eta-} - e^{-i\eta+}],\ e^{-i\eta-} + e^{-i\eta+} - 2e^{-i\eta}\},$$

i.e. the combined integrand for $\{\rho_1, f\rho_0(g\sin\theta)^{-1}(u_1, \mathrm{v}_1)\}$ on $1 < \beta < \alpha_p/\mu$ and $\alpha_p/\mu < \beta$ is

$$- \rho_{z\infty}\,\surd(\nu/\mu)\,(\pi\beta)^{-1}\,\mathrm{d}\beta\,\exp(-\beta t\mu)\,[2\surd\beta - b\surd(\beta+1) - b\surd(\beta-1)]^{-1}$$
$$\times\ \{2\cos\eta - b\cos\eta_- - b\cos\eta_+,\ \surd(b\text{-}1)\,[\cos\eta_- - \cos\eta_+],\ \cos\eta_- + \cos\eta_+ - 2\cos\eta\}$$

which is finite and integrable over the respective ranges, except near the pole $\beta = \alpha_p/\mu > 1$. Here, the denominator $[2\surd\beta - b\surd(\beta+1) - b\surd(\beta-1)]$ has a zero but is analytic as are all other factors. The integral close around the pole was found above; the large contributions on CD and EF, outside the integral around the pole, cancel owing to the denominator's sign change. Hence these components of the solution are finite and integrable; the integrand factor $\exp(-\beta t\mu)$ implies time-decay faster than $\exp(-t\mu)$.

APPENDIX C

Evaluation of inverse Laplace transform for large b ($f = 0$)

The Laplace transforms (u, ρ) are given by (9). In the notation of section 3, if Re(α) is large and positive, then so is the real part of $z_\pm$; the exponentials for $(-\rho_{x\infty}u \pm \lambda\rho)$ decay for large z as required. The argument of $z_\pm$ is thereby defined in $(-\pi/2, \pi/2)$ for all α except the real axis in $\alpha \leq \lambda$. There are branch points at $\alpha = \pm\lambda$ and a pole at $\alpha = 0$. $[(\alpha+\lambda)^{1/2} + (\alpha-\lambda)^{1/2}]$ has no zeros and introduces no other irregularities.

The inverse Laplace transform requires that we evaluate

$$2\pi i\, \rho_{z\infty}^{-1}\, (\rho_1, -\rho_{x\infty}u_1/\lambda)\,(z,t)$$

$$= \nu^{1/2}\int_L \alpha^{-1}[(\alpha+\lambda)^{1/2} + (\alpha-\lambda)^{1/2}]^{-1}\{\exp(-z_+) + \exp(-z_-),\ \exp(-z_-) - \exp(-z_+)\}\, e^{\alpha t}\, d\alpha$$

where the integral path L is from a-iT to a+iT as $T \to \infty$ and a is sufficiently positive for validity. A simplified form of figure 6 (Appendix B) applies if we re-interpret μ as λ. We deform the path from A (a-iT to a+iT) to B (over which the integral is arbitrarily small) giving contributions from pole and the three integral pairs $\int_{C+D}$, $\int_{G+H}$, $\int_{J+K}$, where the lower path of the pair is in the positive sense and the upper path in the negative sense, both being close to the real axis.

Near the pole $\alpha = 0$, let $\alpha = \lambda r e^{i\varphi}$ ($-\pi<\varphi<\pi$; $r<<1$), $\eta \equiv z\,(\lambda/\nu)^{1/2}$. The integrand for

$$\rho_{z\infty}^{-1}\, \{\rho_1, -\rho_{x\infty}u_1/\lambda\}$$

is

$$\surd(\nu/\lambda)\,(2\pi)^{-1}(1-i)^{-1}\{e^{i\eta} + e^{-\eta},\ e^{i\eta} - e^{-\eta}\}\exp(\lambda t r e^{i\varphi})d\varphi \qquad (-\pi<\varphi<0)$$

$$+ \surd(\nu/\lambda)\,(2\pi)^{-1}(1+i)^{-1}\{e^{-i\eta} + e^{-\eta},\ e^{-i\eta} - e^{-\eta}\}\exp(\lambda t r e^{i\varphi})d\varphi \quad (0<\varphi<\pi) \quad + O(r),$$

i.e. the integral is

$$\surd(\nu/\lambda)/2\ \{e^{-\eta} + \cos\eta - \sin\eta,\ \cos\eta - \sin\eta - e^{-\eta}\} \qquad \text{(C.1)}.$$

For the pair CD, let $\beta \equiv -\alpha/\lambda > 1$; then $\surd(\alpha+\lambda) = -i[(\beta-1)\lambda]^{1/2}$ on C and $\surd(\alpha+\lambda) = i[(\beta-1)\lambda]^{1/2}$ on D; also $\surd(\alpha-\lambda) = -i[(\beta+1)\lambda]^{1/2}$ on C and $\surd(\alpha-\lambda) = i[(\beta+1)\lambda]^{1/2}$ on D. The combined integrand over $\beta > 1$ for the CD contribution to $\{\rho_1, -\rho_{x\infty}u_1/\lambda\}$ (z,t) is then

$$\rho_{z\infty}\surd(\nu/\lambda)\,(\beta\pi)^{-1}\,[\surd(\beta+1)+\surd(\beta-1)]^{-1}\exp(-\beta\lambda t)$$

$$\times\ \{-\cos\surd(\beta+1)\eta - \cos\surd(\beta-1)\eta,\ \cos\surd(\beta-1)\eta - \cos\surd(\beta+1)\eta\} \qquad \text{(C.2)},$$

finite and integrable over $\beta > 1$. The factor $\exp(-\beta\lambda t)$ implies time-decay faster than $\exp(-\lambda t)$.

Consider the integration over GHJK combined. For the pair JK, $0 < \beta \equiv \alpha/\lambda < 1$, so $(\alpha-\lambda)^{1/2} = -i[(1-\beta)\lambda]^{1/2}$ on J and $(\alpha-\lambda)^{1/2} = i[(1-\beta)\lambda]^{1/2}$ on K. For the pair GH, $0 < \beta \equiv -\alpha/\lambda < 1$, so $(\alpha-\lambda)^{1/2} = -i[(1+\beta)\lambda]^{1/2}$ on G and $(\alpha-\lambda)^{1/2} = i[(1+\beta)\lambda]^{1/2}$ on H. Also $(\alpha+\lambda)^{1/2} = [(1-\beta)\lambda]^{1/2}$ on G and H. Then writing $\eta \equiv z\,(\lambda/\nu)^{1/2}$, $\eta\pm \equiv \eta(1\pm\beta)^{1/2}$, the combined integrand over $0 < \beta < 1$ for GHJK is

(on J) $\quad \surd(\nu/\lambda)\,\beta^{-1}\,[\surd(1+\beta) - i\surd(1-\beta)]^{-1}\ \{e^{-\eta+} + e^{i\eta-},\ e^{i\eta-} - e^{-\eta+}\}e^{\lambda\beta t}\,d\beta$

(on K) $\quad -\surd(\nu/\lambda)\,\beta^{-1}\,[\surd(1+\beta) + i\surd(1-\beta)]^{-1}\ \{e^{-\eta+} + e^{-i\eta-},\ e^{-i\eta-} - e^{-\eta+}\}e^{\lambda\beta t}\,d\beta$

(on G) $\quad -\surd(\nu/\lambda)\,\beta^{-1}\,[\surd(1-\beta) - i\surd(1+\beta)]^{-1}\ \{e^{-\eta-} + e^{i\eta+},\ e^{i\eta+} - e^{-\eta-}\}e^{-\lambda\beta t}\,d\beta$

(on H) $\quad +\surd(\nu/\lambda)\,\beta^{-1}\,[\surd(1-\beta) + i\surd(1+\beta)]^{-1}\ \{e^{-\eta-} + e^{-i\eta+},\ e^{-i\eta+} - e^{-\eta-}\}e^{-\lambda\beta t}\,d\beta$

$$= i\surd(\nu/\lambda)\,[e^{\lambda\beta t}\,[\surd(1+\beta)\{1,1\}\sin\eta- + \surd(1-\beta)\{e^{-\eta+} + \cos\eta-,\ \cos\eta- - e^{-\eta+}\}]$$

$$- e^{-\lambda\beta t}\,[\surd(1-\beta)\{1,1\}\sin\eta_+ + \surd(1+\beta)\{e^{-\eta-} + \cos\eta_+,\ \cos\eta_+ - e^{-\eta-}\}]]\,\beta^{-1}\,d\beta.$$

We divide the integral in β over (0, 1) into contributions from (0, $Б$) and ($Б$, 1).

For small $\beta > 0$ (and $\eta\sqrt{\beta} \ll 1$) the integrand for $2\pi\rho_{z\infty}^{-1}(\rho_1, -\rho_{x\infty}u_1/\lambda)(z,t)$ is approximately

$$\sqrt{(\nu/\lambda)}\, \beta^{-1}d\beta\, e^{\lambda\beta t}\, [(1+\beta/2-\beta^2/8)\{1, 1\}\sin(1-\beta/2-\beta^2/8)\eta$$
$$+ (1-\beta/2-\beta^2/8)\{\exp(-1-\beta/2+\beta^2/8)\eta + \cos(1-\beta/2-\beta^2/8)\eta,\ \cos(1-\beta/2-\beta^2/8)\eta - \exp(-1-\beta/2+\beta^2/8)\eta\}]$$
$$-\sqrt{(\nu/\lambda)}\, \beta^{-1}d\beta\, e^{-\lambda\beta t}\, [(1-\beta/2-\beta^2/8)\{1,1\}\sin(1+\beta/2-\beta^2/8)\eta$$
$$+ (1+\beta/2-\beta^2/8)\{\exp(-1+\beta/2+\beta^2/8)\eta + \cos(1+\beta/2-\beta^2/8)\eta,\ \cos(1+\beta/2-\beta^2/8)\eta - \exp(-1+\beta/2+\beta^2/8)\eta\}]$$

$$= 2\sqrt{(\nu/\lambda)}\sinh(\lambda\beta t)\,\beta^{-1}\,d\beta\ \ [\{1,1\}[(1-\beta^2/8)\cos\beta\eta/2 - \beta/2\sin\beta\eta/2]\,(\sin\eta^* + \cos\eta^*)$$
$$+ \{1, -1\}[(1-\beta^2/8)\cosh\beta\eta/2 + \beta/2\sinh\beta\eta/2]\,e^{-\eta^*}\,]$$
$$+ 2\sqrt{(\nu/\lambda)}\cosh(\lambda\beta t)\,\beta^{-1}\,d\beta\ \ [\{1,1\}[(1-\beta^2/8)\sin\beta\eta/2 + \beta/2\cos\beta\eta/2]\,(\sin\eta^* - \cos\eta^*)$$
$$+ \{-1, 1\}[(1-\beta^2/8)\sinh\beta\eta/2 + \beta/2\cosh\beta\eta/2]\,e^{-\eta^*}\,]$$

where $\eta^* = (1-\beta^2/8)\eta$, showing that the singularities on the separate paths (G, J, K, H) as $\beta \to 0$ cancel to give a combined integrand that is integrable and that there is no discrete contribution from the apparent pole $\beta = 0$.

We use
$$\int_0^{Б}\cosh(\lambda\beta t)d\beta = (\lambda t)^{-1}\sinh(\lambda Б t)$$
$$\int_0^{Б}\cosh(\lambda\beta t)\,\beta^2\,d\beta = Б^2(\lambda t)^{-1}\sinh(\lambda Б t) - Б(\lambda t)^{-2}\cosh(\lambda Б t) + (\lambda t)^{-3}\sinh(\lambda Б t)$$

etc. and

$$\int_0^{Б}\sinh(\lambda\beta t)\,\beta^{-1}d\beta < \int_0^{b}\sinh(\lambda\beta t)\,\beta^{-1}d\beta + \int_b^{Б}\sinh(\lambda\beta t)\,b^{-1}d\beta$$
$$= \lambda bt + (\lambda bt)^3/(3\times 3!) + (\lambda bt)^5/(5\times 5!) + (\lambda bt)^{-1}[\cosh(\lambda Б t) - \cosh(\lambda bt)] + ..$$
$$\int_0^{Б}\sinh(\lambda\beta t)\,\beta\,d\beta = Б(\lambda t)^{-1}\cosh(\lambda Б t) - (\lambda t)^{-2}\sinh(\lambda Б t)$$
$$\int_0^{Б}\sinh(\lambda\beta t)\,\beta^3\,d\beta = Б^3(\lambda t)^{-1}\cosh(\lambda Б t) - 3Б^2(\lambda t)^{-2}\sinh(\lambda Б t) + 6Б(\lambda t)^{-3}\cosh(\lambda Б t) - 6(\lambda t)^{-4}\sinh(\lambda Б t)$$

etc. Hence the approximate integral over $(0, Б)$ for

$$\rho_{z\infty}^{-1}(\rho_1, -\rho_{x\infty}u_1/\lambda)(z,t)$$

is

$$(\pi\lambda Б t)^{-1}\sqrt{(\nu/\lambda)}\cosh(\lambda Б t)\ [\{1,1\}[(1-Б^2/8)\cos Б\eta/2 - Б/2\sin Б\eta/2]\,(\sin\eta^* + \cos\eta^*)$$
$$+ \{1,-1\}[(1-Б^2/8)\cosh Б\eta/2 + Б/2\sinh Б\eta/2]\,e^{-\eta^*}\ ]$$
$$+ \sqrt{(\nu/\lambda)}\ [\{1,1\}(\sin\eta+\cos\eta) + \{1,-1\}e^{-\eta}]$$
$$\times [\lambda bt/\pi + .. + (\pi\lambda bt)^{-1}(\cosh\lambda Б t - \cosh\lambda bt) + .. - (\pi\lambda Б t)^{-1}\cosh(\lambda Б t)]$$
$$+ (\pi\lambda Б t)^{-1}\sqrt{(\nu/\lambda)}\sinh(\lambda Б t)\ [\{1,1\}[(1-Б^2/8)\sin Б\eta/2 + Б/2\cos Б\eta/2]\,(\sin\eta^* - \cos\eta^*)$$
$$+ \{-1, 1\}[(1-Б^2/8)\sinh Б\eta/2 + Б/2\cosh Б\eta/2]\,e^{-\eta^*}\ ]$$
$$+ \text{terms in higher powers of } (\lambda Б t)^{-1}$$

where now $\eta^* = (1-Б^2/8)\eta$. For large t the growth is bounded by $(\pi\lambda bt)^{-1}\cosh\lambda Б t$ and by $\lambda bt + (\lambda bt)^3/(3\times 3!) + (\lambda bt)^5/(5\times 5!) < 2(\lambda bt)^{-1}\cosh\lambda bt$, i.e. $(\pi\lambda bt)^{-1}\cosh\lambda Б t$ bounds the growth.

For β near 1, let $\beta = 1-\delta$; the integrand for $2\pi\rho_{z\infty}^{-1}(\rho_1, -\rho_{x\infty}u_1/\lambda)(z,t)$ is approximately

$$\surd(\nu/\lambda)\ [e^{\lambda(1-\delta)t}[\surd(2-\delta)\{1,1\}\sin\eta\surd\delta+\surd\delta\ \{e^{-\eta\surd(2-\delta)}+\cos\eta\surd\delta,\ \cos\eta\surd\delta-e^{-\eta\surd(2-\delta)}\}]$$

$$-\,e^{-\lambda(1-\delta)t}[\surd\delta\{1,1\}\sin\eta\surd(2-\delta)+\surd(2-\delta)\{e^{-\eta\surd\delta}+\cos\eta\surd(2-\delta),\ \cos\eta\surd(2-\delta)-e^{-\eta\surd\delta}\}]]\ (1-\delta)^{-1}\,d\beta$$

$$\sim\surd(\nu/\lambda)\ e^{\lambda t}\,e^{-\lambda\delta t}\,[\surd2(1-\delta/4)\{1,1\}\eta(1-\eta^2\delta/6)+\{e^{-\eta\surd2(1-\delta/4)}+1-\eta^2\delta/2,\ 1-\eta^2\delta/2-e^{-\eta\surd2(1-\delta/4)}\}](1+\delta)\surd\delta\ d\beta$$

$$+\,O(e^{-\lambda(1-\delta)t}).$$

We use $\int_Б^1 \delta^{1/2}\,e^{-\lambda\delta t}\,d\beta \;=\; \int_0^{1-Б}\delta^{1/2}\,e^{-\lambda\delta t}\,d\delta \;=\; (\lambda t)^{-3/2}\int_0^{Y} 2y^2\exp(-y^2)\,dy$

$= -(\lambda t)^{-1}(1-Б)^{1/2}\exp[-(1-Б)\lambda t]+(\lambda t)^{-3/2}\pi^{1/2}/2\ \mathrm{erf}(Y)$ where $Y=[(1-Б)\lambda t]^{1/2}$

$\sim\ (\lambda t)^{-3/2}\pi^{1/2}/2$ as $t\to\infty$, any fixed $Б<1$,

and $\int_Б^1 \delta^{n+1/2}\,e^{-\lambda\delta t}\,d\beta \;=\; \int_0^{d}\delta^{n+1/2}\,e^{-\lambda\delta t}\,d\delta + \int_d^{1-Б}\delta^{n+1/2}\,e^{-\lambda\delta t}\,d\delta \;<\; d^{n+3/2} + (1-Б)^{n+3/2}e^{-d\lambda t}$.

Hence for large t and $n\geq1$ we can choose $d=t^{-\alpha}$ $(0.6<\alpha<1)$ so that as $t\to\infty$, any fixed $Б<1$,

$$\int_Б^1 \delta^{n+1/2}\,e^{-\lambda\delta t}\,d\beta \;<\; d^{n+3/2}+(1-Б)^{n+3/2}e^{-d\lambda t}\;\sim\; d^{n+3/2}\;\ll\;(\lambda t)^{-3/2}\pi^{1/2}/2\;\sim\;\int_Б^1\delta^{1/2}\,e^{-\lambda\delta t}\,d\beta.$$

To summarise, at large times the integral in β over $(0, Б) + (Б, 1)$ is dominated by terms of lowest order in $\delta = 1-\beta$ close to $\beta=1$, and

$$\rho_{z\infty}^{-1}\,(\rho_1,\,-\rho_{x\infty}u_1/\lambda)\,(z,t)\;\sim\;[\nu\pi/(4\lambda^2t^3)]^{1/2}\,e^{\lambda t}\,[\surd2\{1,1\}\eta+\{e^{-\eta\surd2}+1,\ 1-e^{-\eta\surd2}\}] \qquad (C.3)$$

where we retain the notations from (2g) and $\eta\equiv z\,(\lambda/\nu)^{1/2}$.

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Figure captions

Figure 1. Definition sketch for context of analysis. An overall context is sketched above, wherein the sloping rectangle is the focus of analysis as expanded in the bottom panel. Vertical lines depict contours of base-state density ρ_∞ (dashed at the bottom where the boundary condition requires at least an adjustment in a bottom boundary layer). The down-slope coordinate is x but the analysis is for base-state density increasing towards shallower water; $\rho_{x\infty} < 0$. The slope angle is θ.

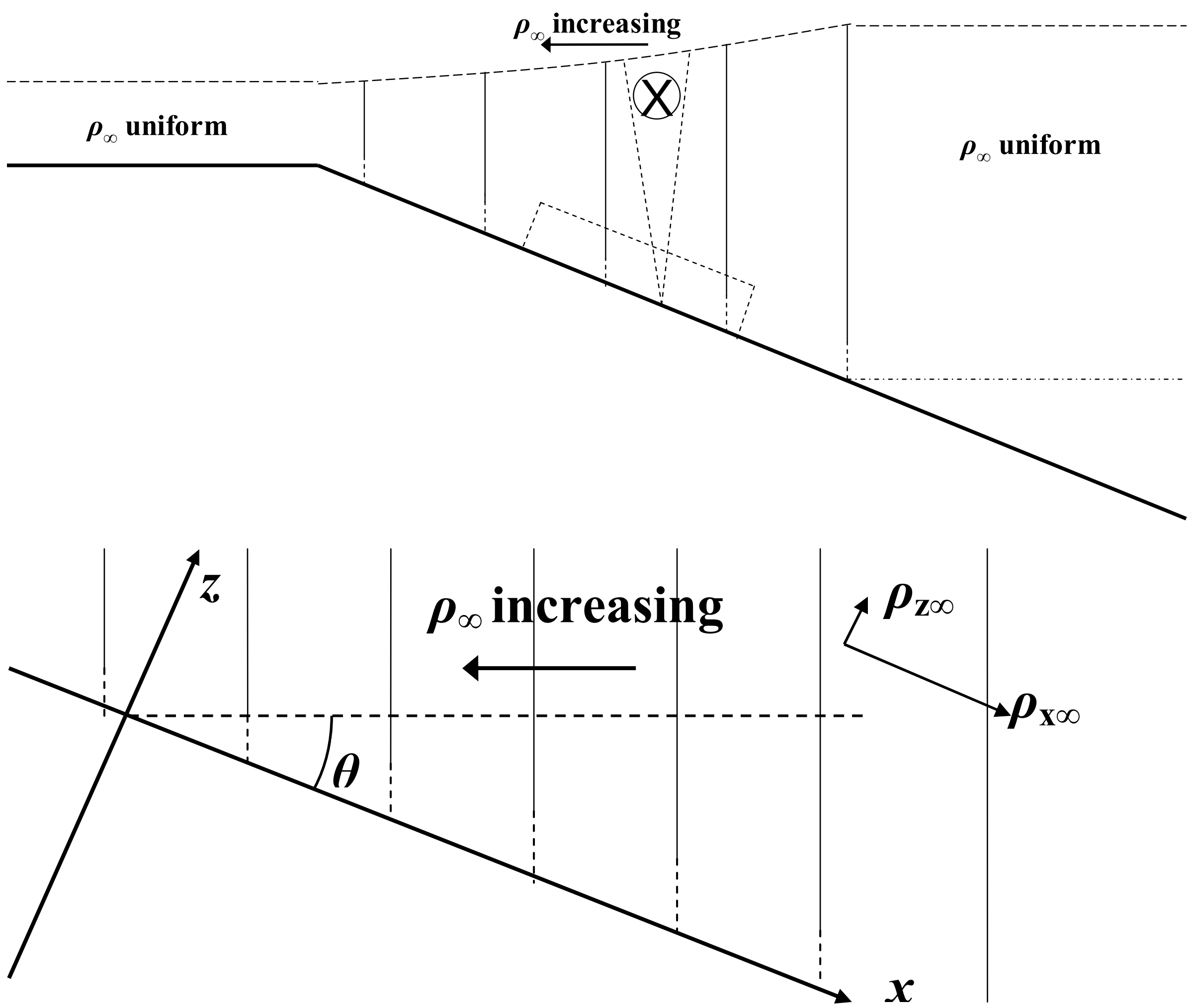
Figure 2. Form of exponentially-growing bottom layer at threshold $b = 1$. ξ is the non-dimensional scaled bottom boundary-layer coordinate. Relative to u and v, ρ is non-dimensionalised by $\rho_{x\infty}/f$ as in (A.2). There is an overall arbitrary factor.

Figure 3. Form of exponentially-growing bottom layer near limit $b = 2^{1/2}$ of analysis validity. ξ is the non-dimensional scaled bottom boundary-layer coordinate. Relative to u and v, ρ is non-dimensionalised by $f\rho_0(g \sin\theta)^{-1}$ as in (B.2). There is an overall arbitrary factor.

Figure 4. Form of fastest-growing part of solution neglecting Coriolis constraint. η is the non-dimensional scaled bottom boundary-layer coordinate. Relative to u, v is small, ρ is non-dimensionalised by $\rho_{x\infty}/\lambda$ as in (C.3). There is an overall arbitrary factor.

Figure 5. Inverse Laplace Transform integral paths when $b = 1$

Figure 6. Inverse Laplace Transform integral paths when $1 < b < 2^{1/2}$



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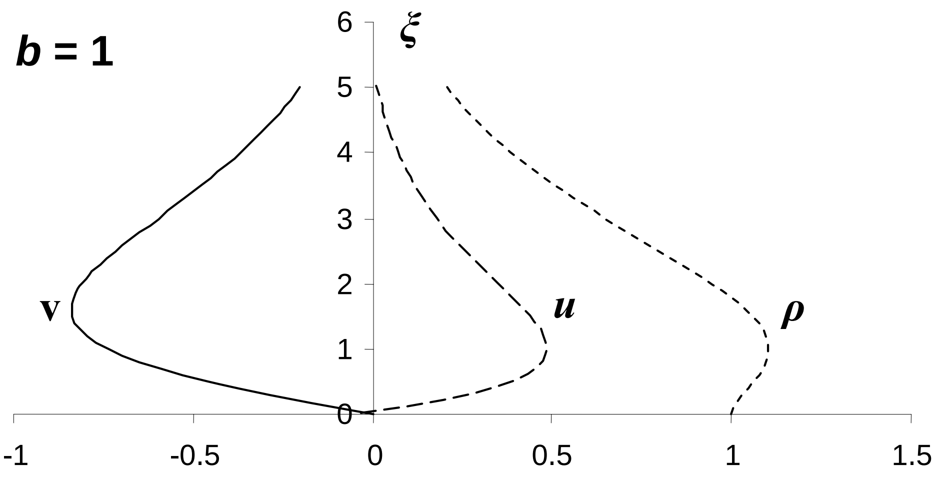

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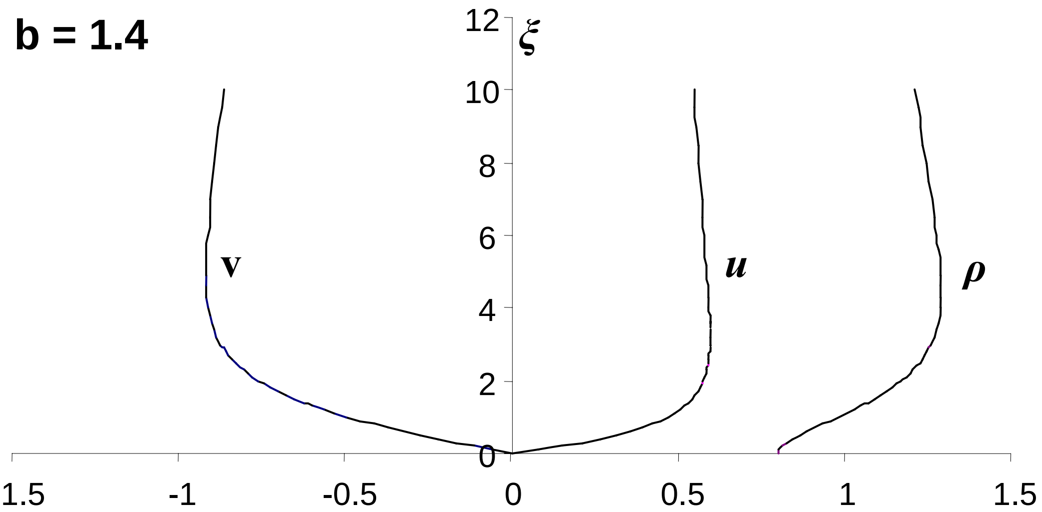

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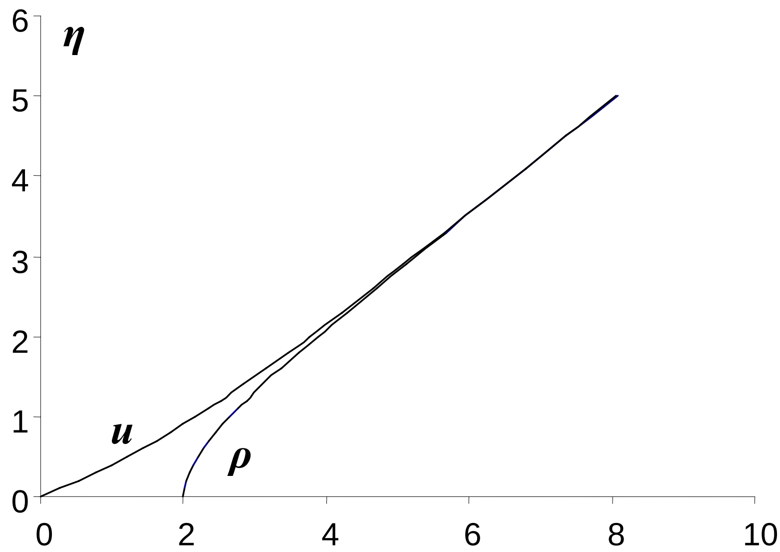


Figure 4. Form of fastest-growing part of solution, as exp(λt), neglecting Coriolis constraint. η is the non-dimensional scaled bottom boundary-layer coordinate. Relative to u, v is small, ρ is non-dimensionalised by $\rho_{x\infty}/\lambda$ as in (C.3). There is an overall arbitrary factor.

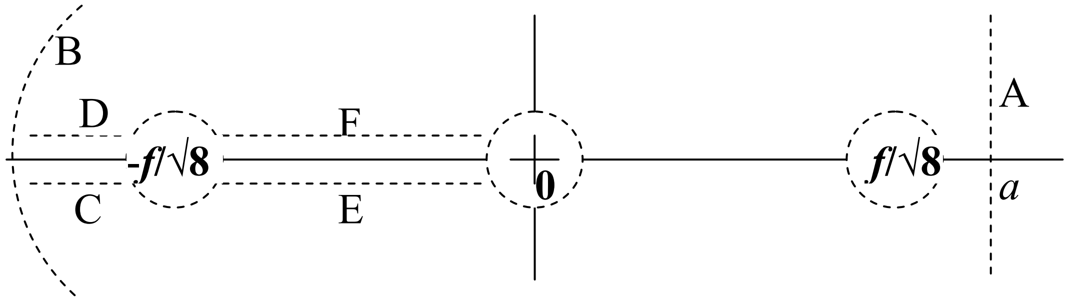
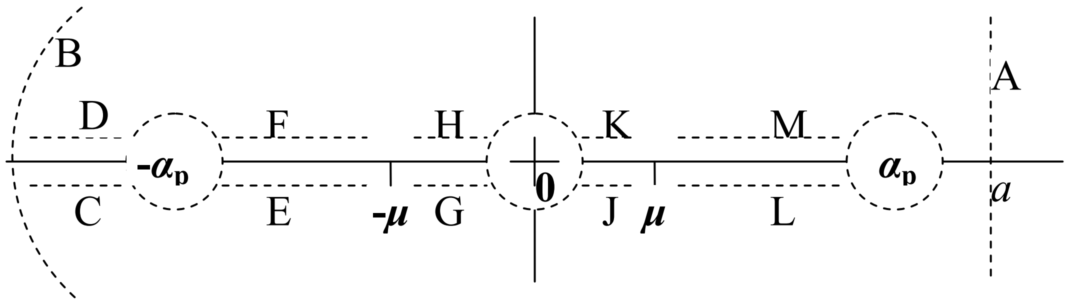


Figure 5. Inverse Laplace Transform integral paths when $b = 1$



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