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Analytical determination of power-law index for the Chapman et al. sandpile (FSOC) analog for magnetospheric activity – a renormalization-group analysis

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Abstract. Recent suggestion and experimental indications that the magnetotail dynamics exhibit self-organized critical behavior have re-motivated interest in sandpile (avalanche) models. Some examples of specific interest for geomagnetic activity have the property that internal avalanches exhibit inverse power law statistics whereas systemwide avalanches have a well-defined mean. Here, we apply the concept of renormalization group to such a model. We demonstrate that invariant analysis based on the renormalization-group theory can explain the power law distribution of energy release by internal avalanches in the large-scale regime of these systems.

1. Introduction

It has been suggested that the Earth's magnetotail may be described by the stochastic behavior of a nonlinear dynamical system near forced and/or self-organized criticality (FSOC) [*Chang*, 1992, 1998, 1999]. There is increasing experimental evidence consistent with this idea. For example, FSOC provides a possible explanation for the intermittent turbulence recently observed in the magnetotail [*Lui et al.*, 1988; *Lui*, 1998; *Angelopoulos et al.*, 1996, 1999]. In-situ magnetic field power spectra in the magnetotail [*Hoshino et al.*, 1994] showed a power law dependence, which is one of the characteristic of the scale-free FSOC behavior. Moreover, analysis of a "burst size" distribution based on AE data also revealed a power law. This may indicate the absence of a characteristic scale within the magnetospheric system [*Consolini,* 1997], or result from FSOC-like scale-free behavior in the turbulent solar wind [*Freeman et al.*, 2000]. *Lui et al.* [2000] used POLAR UVI data to compile a probability distribution for the sizes, and integrated intensities (energy released) of patches of brightness in the nightside aurora, ranging from the smallest scales to substorm breakups. They found that the substorm associated events had a characteristic mean, whereas all other events showed power law probability distributions with index close to -1. Crucially the slope of this index is unchanged by the level of magnetospheric activity.

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The possibility of FSOC in magnetospheric dynamics has re-motivated interest in sandpile (avalanche) models. In such models, the probability distributions of energy released by avalanches, and of avalanche length, should display scale-free, inverse power law statistics that are a characteristic of FSOC [*Bak et al.*, 1987; see also, *Jensen*, 1998]. In contrast to the scale-free behavior exhibited in the internal dynamics of the magnetosphere, substorms, the events of energy release of the magnetotail have well-defined characteristic scales in both intensity and time, a specific class of sandpile models is of particular relevance to the magnetospheric dynamics. This class of models exhibits two scale-free regions in the statistics of its internal avalanches, whereas systemwide events have a well-defined mean [*Chapman et al.,* 1998]. *Chapman et al.* [1999] focused on the statistics of the internal events for large systems, and showed that there is a characteristic broken inverse power law signature in the statistics for energy release. These authors have also established numerically that the two regimes of different power law index correspond to small and large avalanches respectively.

In this study, we utilize the concepts of renormalization group (RG) to analytically determine the power law index for the large internal avalanches in this class of models.

2. Sandpile Model

The avalanche ("sandpile") numerical algorithm on which this study is based is described in more detail in *Chapman et al.* [1999, and references therein]. We review it briefly here. The sandpile is represented by a one-dimensional grid of N equally spaced cells one unit apart, each with sand at height h_j and local gradient $z_j = h_j - h_{j+1}$. The sandpile is assumed to have an "angle of repose" below which it is always stable. The heights h_j and the gradients z_j are measured relative to the values at the angle of repose. Each cell is assigned a critical gradient z_j^*; if the local gradient exceeds this, the sand is redistributed to neighboring cells and iteration produces an avalanche. The critical gradients on each of the N nodes are selected randomly from a top-hat probability distribution $P(z_j^*)$.

The system considered here is edge-driven. Sand is added to cell 1 at a rate g, and the length and time are normalized to this loading rate such that unit volume of sand is added in unit time. As soon as the critical gradient at cell 1 is exceeded, the sand is redistributed. The redistribution is instantaneous, and it conserves the total volume of sand. Sand will propagate to cell 2, and if the local gradient exceeds the critical value there, to cell 3, and so on. Within an avalanche, the sand is instantaneously "flattened" back to the angle of repose (the regime is of uniform gradient). The propagation of an ongoing avalanche from one cell, k, to the next, $k+1$, thus occurs if $h_k - h_{k+1} > z_k^*$. This results in sand being deposited to cell $k+1$ from cells 1 through k, such that the new heights $h_1, h_2, \cdots, h_{k+1}$ are all equal. If now the critical value of the gradient at cell $k+1$ is exceeded, the avalanche continues to $k+2$, and so on. This iterative procedure is repeated until the avalanche reaches a cell where the gradient is below critical. The critical gradients within the post-avalanche region are then regenerated from $P(z_j^*)$. More sand is added at cell 1 until it again becomes unstable, triggering another avalanche. An avalanche may be entirely an internal arrangement of sand, or may continue until it spreads across all N cells of the pile in which case the entire sandpile is emptied and returns to the angle of repose. Note that the system size N has to be finite under these redistribution rules. If N were infinite, we would have to allow for avalanches of infinite length. Such an avalanche would take an infinite amount of time to complete the slide, thereby violating the assumption of an instantaneous redistribution process.

The total energy in the system of the sandpile is:

$$E = \sum_{j=1}^{N} h_j^2. \tag{1}$$

The energy dissipated by an avalanche, ΔE, is just the difference in E before and after the event. Internal avalanches can also be characterized by their length, that is, the number of cells involved in the event. *Chapman et al.* [1999] focused on the statistics of large (5000 cell) systems. Their results showed that for slow fueling rate ($g \ll \langle z^* \rangle$), the normalized probability $P_E(\Delta E)$ exhibits two distinct regimes: $P_E \propto (\Delta E)^{-0.65}$ at small ΔE, but proportional to $(\Delta E)^{-1}$ at large ΔE, with transition at avalanches of size $L \sim 64$ cells. Both the power law indices and the location of the transition in $P_E(\Delta E)$ were shown to be robust against the parameter values of the top-hat distribution $P(z_j^*)$. Of importance to magnetospheric dynamics, the system was also examined under fast fueling $g > \langle z^* \rangle$ and the power law regime for large events was robust under these conditions [*Chapman et al.*, 1999; *Watkins et al.*, 1999].

In this study, we have obtained the statistics of the avalanches based on their size, and found the normalized probability $P_L(L)$ for avalanches of size L. Figure 1 shows $P_L(L)$ for the same data in *Chapman et al.* [1999]; we see that P_L exhibits a power law with index -1 at large L.

A special case of the model has been discussed by *Helander et al.* [1999]. These authors showed that the problem is exactly solvable analytically for the case $P(z_j^*) = \delta(z_j^* - z_0)$. The system exhibits only one regime: $P_E \propto (\Delta E)^{-1}$. It can also be shown that $\Delta E \propto L$ for $L \gg 1$, although it was not mentioned in their paper. One can therefore infer that $P_L \propto L^{-1}$ also in their model. In the following, we shall apply an RG analysis to demonstrate that the $P_L \propto L^{-1}$ behavior at large L is universal to a certain class of sandpile models.

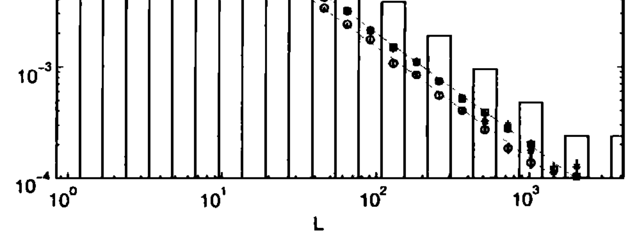

Figure 1. The probability distribution of avalanche lengths for four runs, corresponding to top-hat distributions $P(z_j^*)$ with mean value 1 and different widths. Marked with symbols are cases with widths 0.01 (□), 0.1 (∗) and 1 (○), and marked with bars is the analytically soluble case where the critical gradient takes a single value.

3. Renormalization-group Analysis

The concept of RG involves the search for invariance properties under continued coarse-graining and rescaling of the system [*Wilson and Kogut*, 1974; *Chang et al.*, 1992]. When one coarse-grain the system, and change the rescaling parameter (e.g. the size of the system), transformations involving certain parameters may arise. These transformations may cause some parameters to evolve non-linearly. That would imply certain fixed points in the parameter space. To study the behavior of these parameters near the fixed point, one may linearize the transformations. That would allow one to find power law relations between these parameters and the scaling parameter about the fixed points. Once these power law indices are obtained, the behavior of the system near these fixed points can be understood, and useful information about the system may be extracted, such as the probability distribution of avalanche sizes. Here, we shall demonstrate step-by-step how this procedure can be applied to the sandpile system described above.

Let us consider an ongoing avalanche propagating from cells $k-1$ to k in the sandpile system. We shall define Q_k to be the conditional probability that this avalanche stops at cell k, given that it has already propagated there. The probability for this avalanche to reach $k+1$ is thus $1 - Q_k$, and that for it to reach $k+2$ would be $(1-Q_k)(1-Q_{k+1})$. Now we coarse-grain the system by combining every pair of cells, and renumber the cells (i.e. cells 1 and 2 become $1'$, cells 3 and 4 become $2'$, *etc.*). This process reduces the size of the system, and the length of the avalanches by half ($L \to L/2$). The rearrangement, in particular, combines cells k and $k+1$ into l', where the new label l' numerically equals $(k+1)/2$. The probability for the same avalanche to propagate from l' to the next cell (which consists of the original cell $k+2$) is then $1 - Q_{l'}$

by definition. We can see that if the Q's are to be similarly defined in the rescaled system, we must set

$$Q_{l'} = Q_k + Q_{k+1} - Q_k Q_{k+1}. \tag{2}$$

Because Q_k and Q_{k+1} are associated with adjacent cells, we expect their values to be similar. We thus drop their subscripts, and make them equal. Equation (2) then leads to a transformation

$$R(Q) = 2Q - Q^2, \tag{3}$$

which would make the system look the same after a level of renormalization. Note that we may renormalize the system again. Every subsequent level requires the rescaling $L \to L/2$, and the transformation $Q \to R(Q)$, in order to make the system look the same. The transformation supports two fixed points: $Q_a = 0$, and $Q_a = 1$. The case $Q = 1$ means that an avalanche will always stop propagating once it reaches a particular cell; this case is irrelevant to the sandpile system. The case $Q = 0$, on the other hand, corresponds to the situation where an avalanche will always go on to the next cell. We shall argue that this case is relevant to the sandpile system by showing that Q_k is small as k becomes large.

For an ongoing avalanche to propagate from cells $j-1$ to j, the average height of the sand from 1 to $j-1$ must exceed $h_j + z_j^*$. Therefore, for an avalanche to reach k, the following conditions must be satisfied:

$$\sum_{j=1}^{n} j z_j > n z_n^*, \; n = 1, \ldots, k-1. \tag{4}$$

For this avalanche to stop at k, it requires:

$$\sum_{j=1}^{k} j z_j \leq k z_k^*. \tag{5}$$

Condition (5) can be combined with the $n = k-1$ case in (4) to determine the range of z_k that would satisfy the inequalities:

$$z_k \leq z_k^* - \frac{1}{k}\sum_{j=1}^{k-1} j z_j < z_k^* - \frac{k-1}{k} z_{k-1}^*. \tag{6}$$

Note that the middle or the right-hand side of (6) may be negative, depending on the random choices of z_k^* and z_{k-1}^*. In that case, condition (6) can never be satisfied (because $z_k \geq 0$). Otherwise, we can take the ensemble average of the right-hand side of (6) to estimate an upper bound for the range of z_k that satisfies the condition. This estimated upper bound is $\langle z^* \rangle / k$, where $\langle z^* \rangle = \langle z_k^* \rangle = \langle z_{k-1}^* \rangle$. Thus, the range of z_k that contributes to the probability Q_k decreases as k becomes larger. We therefore conclude that Q_k is small for large k. Based on this argument, we expect that if N is sufficiently large, there exists a cell $K < N$ such that for $k > K$, Q_k is in the regime governed by the fixed point $Q_a = 0$.

Now we renormalize the system, and study the behavior near $Q = 0$. For each step of transformation, L and Q evolve as:

$$L' = L/2, \text{ and } Q' = 2Q - Q^2. \tag{7}$$

Near $Q = 0$, the last term in (7) is small. Thus, one can show that under the transformation,

$$Q \sim 1/L + \mathcal{O}(1/L^2). \tag{8}$$

Or QL is an invariant under the RG transformation for large L and small Q. Note that this result is true only for avalanches of large L, where Q is close to the fixed point. In this regime, the probability that an avalanche which has reached a cell k will go no further is inversely proportional to the length of the avalanche itself. Of course, if the avalanche stops at cell k, its length will be k. We can thus express the result as $Q_k \propto k^{-1} + \mathcal{O}(k^{-2})$. Now let us consider not only those avalanches that reach k, but avalanches of all lengths. An arbitrary avalanche that stops at $k > K$ must have propagated through $K, K+1, \ldots, k-1$. Hence, for $k > K$, we can normalize P_L by the factor $(1 - Q_1)\ldots(1 - Q_{K-1})$, and write:

$$\begin{aligned} P_L(L = k) &\propto (1 - Q_K)(1 - Q_{K+1})\ldots(1 - Q_{k-1})Q_k \\ &\propto Q_k + \mathcal{O}(Q^2) \\ &\propto k^{-1} + \mathcal{O}(k^{-2}), \end{aligned} \tag{9}$$

as all the Q's in Eq. (9) are much smaller than unity. This result is equivalent to

$$P_L(L) \propto L^{-1} + \mathcal{O}(L^{-2}). \tag{10}$$

Therefore, for small Q_k (or large L, see Fig. 1), our RG analysis has reproduced to the leading order the scaling law obtained by large-scale avalanche model calculations.

The analysis above, however, has yet to explain the power law behavior for P_E. Here we look for a relationship between ΔE and L for large-scale avalanches, and then determine the power law between P_E and ΔE based on the results above. We recognize that the energy of the system, given by Eq. (1), scales as the system size N. When we renormalize, the length of the system is halved. One can thus write the renormalized energy as $E' = E/2$. To find ΔE, we simply take the difference in total energy before and after the avalanche. Thus, $\Delta E' = (\Delta E)/2$. We recognize that ΔE and L are reduced by the same factor under renormalization. Thus, we establish that $\Delta E \propto L$, or equivalently, the factor $(\Delta E)/L$ is an invariant in the regime of large L. With Eq. (10), we can conclude that there is a regime for large ΔE where

$$P_E(\Delta E) \propto (\Delta E)^{-1} + \mathcal{O}((\Delta E)^{-2}). \tag{11}$$

We note that the analysis above is quite robust. Suppose we use a slightly different coarse-graining process: Instead of combining every 2 cells in the system, we may perform the procedure with every 3, 4, ...or M cells. Provided M is not too large, such that the Q's are essentially the same, the renormalization transformation for Q is then:

$$R_M(Q) = 1 - (1 - Q)^M. \tag{12}$$

There are M fixed points for the transformation R_M: $Q_a = 0$ and $Q_a = 1$ are still fixed points, while the other

$M-2$ do not lie within the real line between 0 and 1. These other fixed points are thus irrelevant. Near $Q=0$, the linear behavior discussed above is retained. One can show that now $Q' = MQ$ and $L' = L/M$. Therefore, QL is still an invariant under the RG transformations in the regime of large L and small Q. Similarly, $E' = E/M$ under the renormalization, implying $(\Delta E)/L$ to be an invariant in this regime. Therefore, the power law behaviors that we found earlier can be obtained from a more general analysis based on the RG theory.

We should emphasize that the RG analysis above is quite general. It applies not only to one particular sandpile model. In fact, Eq. (10) is true for any finite model where $Q \ll 1$ near the edge. Hence, the analysis is also applicable to models of higher dimensions, as long as the property involving Q is satisfied.

4. Conclusion

We have demonstrated that the concept of RG can be applied to a class of sandpile (avalanche) models. The FSOC behavior demonstrated by this class of systems is of particular interest in the study of the magnetospheric dynamics; statistics of internal avalanches in the models exhibits two scale-free regions, whereas that of systemwide events has a well-defined mean. RG analysis has allowed us to explain the power-law relations that characterize the large-scale regime of internal avalanches in the sandpile systems. The analysis is quite general, and is applicable to other sandpile models, or perhaps even more complicated avalanche systems that exhibit FSOC. In more general applications, recursion equations such as (3) and (12) may include more than one parameter and more complex nonlinearity, leading to power-law relations other than -1.

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