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**PROBABILITY-WEIGHTED  
MOMENTS ESTIMATORS FOR  
TCEV PARAMETERS**

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## Introduction

The Two Component Extreme Value (TCEV) distribution was advocated by Rossi et al. (1984) for use in flood frequency analysis. They presented a procedure, modified by Fiorentino et al. (1986), for estimating site and regional parameters by maximum likelihood. This note describes an alternative estimation procedure based on probability weighted moment (PWM). Details of the TCEV distribution, its fit to real data and its statistical properties, are given in Rossi et al. (1984), Arnell and Beran (1987) and Arnell and Gabriele (1988), and are not repeated here.

The TCEV distribution of annual maxima has the distribution function:

$$F(x) = \exp[-\lambda_1 e^{-x/\theta_1} - \lambda_2 e^{-x/\theta_2}] \quad x \geq 0 \quad \dots (1)$$

or, in the standardised regional case

$$F(y) = \exp[-e^{-y} - \lambda_* e^{-y/\theta_*}] \quad \dots (2)$$

where

$$\begin{aligned} \theta_* &= \theta_2/\theta_1 \\ \lambda_* &= \lambda_2/(\lambda_1^{1/\theta_*}) \end{aligned} \quad \dots (3)$$

and

$$y = \frac{x - \theta_1' \ln \lambda_1'}{\theta_1'} \quad \dots (4)$$

$\theta_1'$  and  $\lambda_1'$  are site parameters, whilst  $\theta_1$ ,  $\theta_2$ ,  $\lambda_1$  and  $\lambda_2$  are regional parameters.

## Probability-weighted Moments

Probability-weighted moments are defined as (Greenwood et al, 1979)

$$M_{p,r,s} = E [x^p F(x)^r (1 - F(x))^s] \quad \dots (5)$$

where  $p$ ,  $r$  and  $s$  are real numbers. PWMs for the TCEV distribution can be derived with  $p = 1$  and  $s = 0$  in equation (5) (Beran et al. 1986):

$$\beta_r = \text{PWM}_r^{(1)} + \frac{\theta_1}{r+1} T_r \quad \dots (6)$$

where

$$\text{PWM}_r^{(1)} = \frac{\theta_1}{r+1} \{ \gamma + \ln \lambda_1 + \ln(r+1) \} \quad \dots (7)$$

and

$$T_r = \sum_{j=1}^{\infty} (-1)^{j-1} \lambda_1^j (r+1)^{j-1/\theta_1} \Gamma(j/\theta_1) / j! \quad \dots (8)$$

$\gamma$  is the Euler number.

Two different procedures for estimating TCEV parameters using PWMs have been developed, based on L-moments (Hosking, 1986) and central PWMs.

## PWM Estimators Based on L-moments

U-statistics are linear combinations of PWMs (Hosking, 1986), and four define completely the TCEV:

$$L_1 = \beta_0 = \theta_1 [\gamma + \ln \lambda_1 + T_0] \quad \dots (9a)$$

$$L_2 = 2\beta_1 - \beta_0 = \theta_1 [\ln 2 + D_1] \quad \dots (9b)$$

$$L_3 = 6(\beta_2 - \beta_1) + \beta_0 = \theta_1 [\ln 9/8 + 2D_2 - D_1] \quad \dots (9c)$$

$$\begin{aligned} L_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0 \\ &= \theta_1 [\ln 2^{16}/3^{10} + 5D_3 - 5D_2 + D_1] \quad \dots (9d) \end{aligned}$$

where  $T_r$  is as above and

$$D_r = T_r - T_{r-1}$$

These equations could be solved to yield the four TCEV parameters, but an easier solution is obtained using L-moments, which are ratios of the U-statistics (Hosking, 1986).

$$\tau = L_2/L_1 = (\ln 2 + D_1)/(\gamma + \ln \lambda_1 + T_0) \quad \dots (10a)$$

$$\tau_3 = L_3/L_2 = (\ln 9/8 + 2D_2 - D_1)/(\ln 2 + D_1) \quad \dots (10b)$$

$$\tau_4 = L_4/L_2 = (\ln 2^{16}/3^{10} + 5D_3 - 5D_2 + D_1)/(\ln 2 + D_1) \quad \dots (10c)$$

$\tau_3$  and  $\tau_4$  are dimensionless, and are measures of skewness and kurtosis.  $\theta_*$  and  $\lambda_*$  can be obtained using just  $\tau_3$  and  $\tau_4$  by solving the following two equations:

$$f_1 = \frac{\ln 9/8 + 2D_2 - D_1}{\ln 2 + D_1} - M3 = 0 \quad \dots (11a)$$

and

$$f_2 = \frac{\ln 2^{16}/3^{10} + 5D_3 - 5D_2 + D_1}{\ln 2 + D_1} - M4 = 0 \quad \dots (11b)$$

where M3 and M4 are the sample estimates of the two L-moments  $\tau_3$  and  $\tau_4$ . Equations (11a) and (11b) can be solved using the Newton-Raphson method:

$$\begin{aligned} u^{(k+1)} &= u^{(k)} + du & ; & & t^{(k+1)} &= t^{(k)} + dt \\ u &= \theta_* \ln \lambda_* \\ t &= 1/\theta_* \end{aligned} \quad \dots (12)$$

where

$$du = - \left( \frac{df_2}{dt} f_1 - \frac{df_1}{dt} f_2 \right) / \left( \frac{df_1}{du} \frac{df_2}{dt} - \frac{df_1}{dt} \frac{df_2}{du} \right) \quad \dots (13a)$$

$$dt = - \left( - \frac{df_2}{du} f_1 + \frac{df_1}{du} f_2 \right) / \left( \frac{df_1}{du} \frac{df_2}{dt} - \frac{df_1}{dt} \frac{df_2}{du} \right) \quad \dots (13b)$$

The derivatives are

$$\left. \begin{aligned} \frac{df_1}{du} &= \frac{1}{g} [2D_{2,u} - (1+h_1)D_{1,u}] \\ \frac{df_1}{dt} &= \frac{1}{g} [2D_{2,t} - (1+h_1)D_{1,t}] \\ \frac{df_2}{du} &= \frac{1}{g} [5D_{3,u} - 5D_{2,u} + (1-h_2)D_{1,u}] \\ \frac{df_2}{dt} &= \frac{1}{g} [5D_{3,t} - 5D_{2,t} + (1-h_2)D_{1,t}] \end{aligned} \right\} \quad \dots (14)$$

where

$$\begin{aligned}
 D_{r,u} &= \frac{dT_r}{du} - \frac{dT_{r-1}}{du} ; & D_{r,t} &= \frac{dT_r}{dt} - \frac{dT_{r-1}}{dt} \\
 \frac{dT_r}{du} &= t \sum_{j=1}^{\infty} \frac{(-1)^{j-1} e^{jut} \Gamma(jt) (r+1)^j (1-t)}{(j-1)!} \\
 \frac{dT_r}{dt} &= [u - \ln(r+1)] \sum_{j=1}^{\infty} \frac{(-1)^{j-1} e^{jut} \Gamma(jt) (r+1)^j (1-t)}{(j-1)!} \\
 &+ \sum_{j=1}^{\infty} e^{jut} \frac{\Gamma(jt) \psi(jt)(r+1)^j (1-t)}{(j-1)!} \dots (15)
 \end{aligned}$$

$$g = \ln 2 + D_1 ;$$

$$h_1 = \frac{\ln 9/8 + 2D_2 - D_1}{g} ;$$

$$h_2 = \frac{\ln 2^{16}/3^{10} + 5D_3 - 5D_2 + D_1}{g}$$

A regional TCEV distribution can be fitted using the regional average of the sample estimates of M3 and M4 (weighted by record length), giving regional  $\theta_*$  and  $\lambda_*$ , and hence a regional frequency curve in terms of the normalised variate

$$Y = \frac{x - \theta_1' \ln \lambda_1'}{\theta_1'} \dots (16)$$

$\theta_1'$  and  $\lambda_1'$  are site parameters. This regional curve can be converted into a curve showing the T-year flood as a multiple of the mean  $\bar{x}$  using a regional average  $\lambda_1'$ . Site estimates of  $\lambda_1'$  are obtained by solving equation (10a) with site estimates M2 of  $\tau$ , and these site values can be averaged to give  $\bar{\lambda}_1$ . Alternatively a regional average M2 can be used to determine  $\bar{\lambda}_1$ .

Constrained TCEV distributions can be easily fitted to individual samples,

where  $\theta_*$ ,  $\lambda_*$  and  $\bar{\lambda}_1$  are fixed.  $\theta_1'$  can be obtained explicitly from Equation (9a), and  $\theta_2'$  and  $\lambda_2'$  are determined from  $\theta_*$ ,  $\lambda_*$ ,  $\theta_1'$  and  $\bar{\lambda}_1$ .

## PWM Estimators Using Central PWM's

This version of the PWM estimation procedure was developed by Fabio Rossi, and uses central PWMs defined as

$$\beta_r' = E[(x - \beta_0)^r F^r] \quad \dots (17)$$

Central PWMs are linear functions of the more usual non-central PWMs  $\beta_r$ :

$$\beta_r' = \beta_r - \frac{\beta_0}{r+1}$$

The first two central PWMs are

$$\beta_0' = 0 \quad \dots (18a)$$

$$\beta_1' = \beta_1 - \frac{\beta_0}{2} = \theta_1[\ln 2 + T_1 - T_0]/2 \quad \dots (18b)$$

$$\beta_2' = \beta_2 - \frac{\beta_0}{3} = \theta_1[\ln 3 + T_2 - T_0]/3 \quad \dots (18c)$$

$$\beta_3' = \beta_3 - \frac{\beta_0}{4} = \theta_1[\ln 4 + T_3 - T_0]/4 \quad \dots (18d)$$

(Note that  $L_2 = 2\beta_1'$ )

$$L_3 = 6(\beta_2' - \beta_1') \quad \dots (19)$$

and

$$L_4 = 20\beta_3' - 30\beta_2' + 12\beta_1' \quad )$$

TCEV parameters  $\theta_*$  and  $\lambda_*$  are estimated using ratios of these central PWMs:

$$\frac{3}{2} (\beta_2'/\beta_1') = \frac{\ln 3 + T_2 - T_0}{\ln 2 + T_1 - T_0} \quad \dots (20a)$$

$$2(\beta_3'/\beta_1') = \frac{\ln 4 + T_3 - T_0}{\ln 2 + T_1 - T_0} \quad \dots (20b)$$

and, as before, the solution can be found using a Newton-Raphson method. In the regional case  $\theta_*$  and  $\lambda_*$  are estimated using regional averages of sample values  $B_2/B_1$  and  $B_3/B_1$ . Equations 10b and 10c differ from 20a and 20b only in the numerator, and the central PWM ratios are therefore related to the L-moments by

$$\frac{3}{2} (\beta_2'/\beta_1') = \tau_3 \frac{\ln 3 + T_2 - T_0}{\ln 9/8 + 2D_2 - D_1} \quad \dots (21a)$$

$$2(\beta_3'/\beta_1') = \tau_4 \frac{\ln 4 + T_3 - T_0}{\ln 2^{16}/3^{10} + 5D_3 - 5D_2 + D_1} \quad \dots (21b)$$

Regional average values of  $\lambda_1'$  can be obtained by averaging the site estimates computed from:

$$2(\beta_1'/\beta_0) = \frac{\ln 2 + T_1 - T_0}{\ln \lambda_1 + T_0 + \gamma} \quad \dots (22)$$

where the site estimate  $B_1'/B_0$  is used to estimate  $\beta_1'/\beta_0$ . Equation (22) is equivalent to Equation 10(a). Site estimates of  $\theta_1'$  can be obtained from Equation (18b) using site estimates of the central PWMs.

## Practical Application

Both versions of the PWM estimator for the regional TCEV distribution are much simpler to apply than the maximum likelihood procedure currently used.

Firstly, there is no need to start with at-site estimates of  $\theta_1'$  and  $\lambda_1'$ , since the dimensionless ratios  $M_3$  and  $M_4$  (and  $B_3/B_1$  and  $B_2/B_1$ ) describe both the distribution of  $x$  and its linear transform  $Y$ . This is a major advantage over the maximum likelihood procedure, as it has been found that estimation of  $\theta_1'$  and  $\lambda_1'$  is often the most difficult stage. The estimation of the parameters  $\theta_*$  and  $\lambda_*$  is also much quicker, as there is no need to compute the values of the likelihood function and its derivatives over a large sample, and it is, of course, not necessary to repeat the process with new estimates of at-site  $\theta_1'$  and  $\lambda_1'$ .

The studies so far have used unbiased estimates of PWMs:

$$\hat{\beta}_r = n^{-1} \sum_{j=1}^n \frac{(j-1)(j-2) \dots (j-r)}{(n-1)(n-2) \dots (n-r)} x_j \quad \dots (23)$$

Regional TCEV distributions were fitted to data from 58 UK stations using both the L-moments and central PWM estimators, and compared with a distribution fitted by maximum likelihood. Parameter estimates are shown in Table 1, and Table 2 shows regional quantiles in terms of both reduced variate  $Y$  and the ratio of the T-year flood to the mean  $\bar{x}$ .

*Table 1 Regional TCEV parameters, UK data*

	TCEV-PWM L-moments	TCEV-PWM central	TCEV-MLE
$\theta_*$	6.0253	6.0533	4.4542
$\lambda_*$	0.0117	0.0116	0.0293
$\lambda_1$	16.0002	15.9977	18.4146
outlier probability	0.011	0.011	0.026

*Table 2 Regional frequency curves, TCEV distribution*

Return period	TCEV-PWM L-moments	TCEV-PWM central	TCEV-MLE
(a) $Y_T$ quantiles			
20	3.117	3.116	3.291
50	4.242	4.239	4.624
100	5.268	5.265	6.011
500	10.722	10.725	11.965
1000	14.837	14.859	15.043
(b) $x_T/\bar{x}$			
20	1.725	1.724	1.719
50	2.054	1.053	2.088
100	2.354	2.354	2.473
500	3.951	3.953	4.122
1000	5.156	5.163	4.975



It is clear from the tables that the two PWM estimators are very similar, and this is to be expected given that they are linearly related: in fact, they should be identical. The PWM estimators give very different parameter estimates to those from the maximum likelihood fit, although the difference between the quantiles is not so great.

During the fitting it was found that the choice of starting estimates of  $\theta_*$  and  $\lambda_*$  for the Newton-Raphson procedure was important, and that with some starting points the algorithm did not converge. However, when it did converge it always reached the same solution. The problem was eliminated by incorporating a random starting point, with new starting points repeatedly chosen until the algorithm converged.

Some simulation experiments have been conducted with the L-moment version of the PWM estimator, using synthetic regions with 40 stations each of 40 years drawn from TCEV distributions based on fits to UK and Italian data (to enable comparisons with results of the TCEV-MLE experiments presented in Arnell and Gabriele (1988)). The most important finding was that for a high proportion of synthetic regions the TCEV-PWM algorithm failed to reach a solution - the proportions failing were 40.8% for the UK-based parent and 7.4% for the Italian-based parent (the TCEV-MLE procedure reached a solution for all of the synthetic regions). These failures were not (on the whole) due to inadequate starting points, but reflect the limited range of combinations of  $\tau_3$  and  $\tau_4$  feasible with the TCEV distribution. Figure 1 shows  $\theta_*$  and  $\lambda_*$  plotted against L-moments  $\tau_3$  and  $\tau_4$ , and it is clear that there is a large number of combinations of  $\tau_3$  and  $\tau_4$  which cannot be accommodated by the TCEV distribution. It was found that the vast majority of synthetic regions which did not yield a solution had sample estimates of  $M_3$  and  $M_4$  which fell outside the 'feasible space' of the TCEV distribution. This has also been found with some real-world flood and rainfall data, and suggests a significant weakness in the TCEV-PWM procedure.

D. Jones (pers. comm.) has developed approximations for the bounds implied in Fig. 1. The upper bound is given by:

$$\tau_4' = \frac{\tau_3 (\ln(2^{15}/3^{10}) + \ln 3^{12}/2^{19})}{\ln(3^2/2^4)} \quad \dots (24)$$

and the lower bound is defined by:

$$\tau_3' = \frac{2 [\ln 3/2 + E_1(3\lambda_*) - E_1(2\lambda_*)]}{\ln 2 + E_1(2\lambda_*) - E_1(\lambda_*)} - 1 \quad \dots (25)$$

$$\tau_4' = \frac{5 [\ln 8/9 + E_1(4\lambda_*) - 2E_1(3\lambda_*) + E_1(2\lambda_*)]}{\ln 2 + E_1(2\lambda_*) - E_1(\lambda_*)} + 1$$

where  $E_1(\cdot)$  is the exponential integral. The TCEV is only defined for  $\tau_3$  and  $\tau_4$  above the line defined by  $(\tau_3', \tau_4')$ .

Table 3 shows the mean and standard deviation of estimates of dimensionless  $x_T/\bar{x}$  quantiles for both parents and for both ML and PWM estimates: it must be remembered that a solution was not found for all PWM repetitions.

The simulation results show that the TCEV-PWM method tends to produce lower growth curves (except for higher quantiles with the Italian parent and low quantiles with the UK parent) with a variability similar to that of the TCEV-MLE procedure (although variability is less with TCEV-PWM at high quantiles with the UK parent). Overall, the TCEV-PWM algorithm does not seem to show a dramatic improvement over the TCEV-MLE procedure, and any reduced bias tends to be offset by higher standard deviation.

*Table 3 Mean and standard deviation of growth factors  $x_T/\bar{x}$ , 40 stations with 40 years each, 500 repetitions.*

	return period				
	20	50	100	500	1000
(a) TCEV-1 (UK-based parent)					
true	1.645	1.981	2.334	3.793	4.532
	mean growth factors				
TCEV-MLE	1.636	1.968	2.310	3.654	4.532
TCEV-PWM	1.658	1.979	2.277	3.388	4.062
	standard deviation				
TCEV-MLE	0.020	0.054	0.122	0.536	0.752
TCEV-PWM	0.024	0.056	0.106	0.423	0.691
(b) TCEV-2 (Italian-based parent)					
true	2.037	2.721	3.324	4.794	5.431
	mean growth factors				
TCEV-MLE	2.032	2.689	3.260	4.649	5.251
TCEV-PWM	2.017	2.644	3.221	4.717	5.375
	standard deviation				
TCEV-MLE	0.044	0.095	0.170	0.407	0.515
TCEV-PWM	0.048	0.098	0.160	0.443	0.596

## Summary

From an operational point of view, the TCEV-PWM algorithm is much quicker and easier to apply than the TCEV-MLE algorithm, but restrictions in the range of feasible combinations of PWM ratios mean that the algorithm does not always converge: this has been found with both real and synthetic data. The PWM and ML estimators can give quite different parameter estimates when applied to the same data set, although estimated quantiles are less different.

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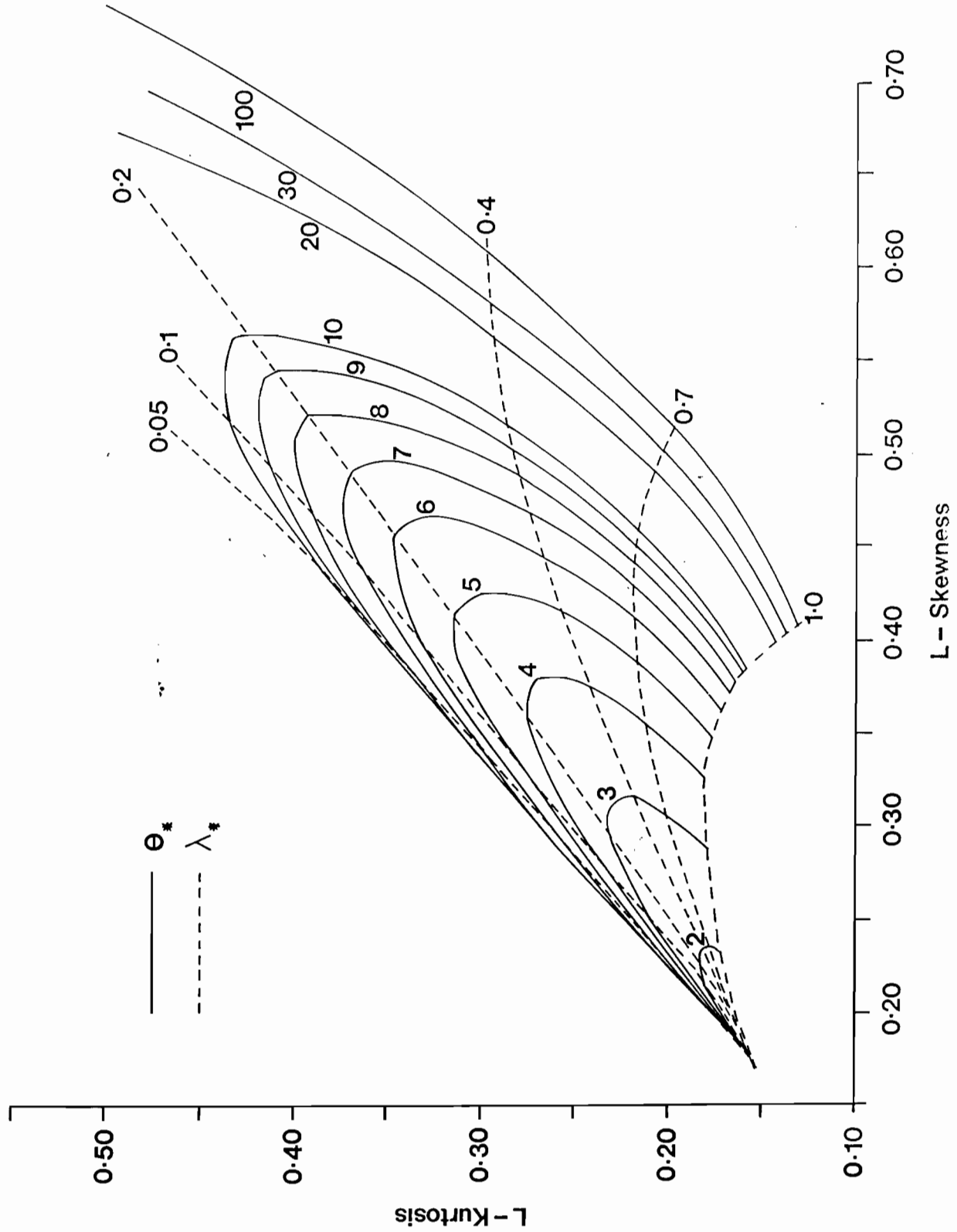
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Figure 1. Variation of  $\theta_*$  and  $\lambda_*$  with L-skewness and L-kurtosis.



The demand for long-term scientific capabilities concerning the resources of the land and its freshwaters is rising sharply as the power of man to change his environment is growing, and with it the scale of his impact. Comprehensive research facilities (laboratories, field studies, computer modelling, instrumentation, remote sensing) are needed to provide solutions to the challenging problems of the modern world in its concern for appropriate and sympathetic management of the fragile systems of the land's surface.

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