I.O.S.

CONFIDENCE LIMITS FOR EXTREME VALUE STATISTICS

by
P. G. CHALLENOR

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NATURAL ENVIRONMENT RESEARCH COUNCIL
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ABSTRACT

The derivation of confidence limits for the Fisher-Tippett 1 distribution is discussed and formulae given in Lawless (1974) are used to produce tables of percentage points. Using these tables confidence limits can be produced for the scale and location parameters of the distribution as well as various return values. In addition tables are given to produce prediction limits on the largest in a future sample.

INTRODUCTION

The Fisher-Tippett Type 1 (FT 1) distribution is defined by

$$P(X<x) = \exp(-\exp(-(x-A)/B)) \quad B>0$$

where A and B are location and scale parameters respectively.

This distribution is known under a wide variety of names, some of the more usual ones are the extreme value distribution, the first extreme value distribution and the Gumbel distribution. Confusingly these names are also sometimes applied to the distribution of -X.

The FT 1 distribution is one of the three asymptotic distributions of maxima initially derived by Fisher and Tippett (1928). The other two involve an extra parameter and are bounded, and are not so widely used for the analysis of extremes. One of the statistics usually calculated from geophysical extreme data is the return value, most often the 50 or 100 year return value e.g. Hardman et al (1973). Depending upon the sampling rate, these return values are equivalent to certain percentiles of the parent distribution. In this note all reference will be to percentiles so that the results can easily be transferred from one sampling rate to another.

The results presented here, computed from formulae given by Lawless (1974), enable approximate confidence limits to be calculated for A, B, the median and the 98th and 99th percentiles. In addition to these confidence limits, tables are given which can be used to produce prediction limits for the largest in a future sample.

MAXIMUM LIKELIHOOD ESTIMATION

The confidence limits given here are based upon the maximum likelihood estimates of A and B. However, similar methods could be used to derive confidence intervals for any estimators having certain invariance properties, for example the best linear invariant estimators could be used, for details see Lawless (1978).

It is not possible to solve the likelihood equations for the FT 1 distribution analytically. However they can be reduced to the following pair of equations.

$$\hat{B} = (\sum_{i=1}^{n} x_i)/n - (\sum_{j=1}^{n} x_j e^{-x_j/\hat{B}})/(\sum_{j=1}^{n} e^{-x_j/\hat{B}})$$

$$\hat{A} = -\hat{B} \log (\frac{1}{n}\sum_{j=1}^{n} e^{-x_j/\hat{B}})$$

where x_j $j=1(1)n$ is the j^{th} member of a sample, size n.

The first equation can be solved iteratively for $\hat{B}$ and this can then be inserted into the second to produce $\hat{A}$. Various approximations to $\hat{A}$ and $\hat{B}$ have been suggested and a full account is given in Johnson and Kotz (1970). (A FORTRAN subroutine to find the maximum likelihood estimates, SUBROUTINE MEST, is available in LIBRARY/IOS.

CONFIDENCE INTERVALS

A 100α% confidence interval about a parameter θ is an interval whose end points are statistics such that the probability that this interval covers θ is α.

Note that the value of θ is fixed and it is the end points of the confidence interval that are functions of the data and vary from sample to sample. For example, if we produce 90% confidence intervals from 10 samples we would expect 9 out of the 10 to include the fixed value of θ.

The above definition does not specify how the remaining probability (1-α) is distributed at each end of the interval. There is an infinity of possibilities, so some criterion must be chosen before we can select an interval. One of these would be to take the shortest possible interval. While this is possibly the ideal it is not normally used because of the vast amount of computation necessary. Another possibility would be to have the interval centred on some point estimate of the parameter. The most usual criterion is the one used here, that is to have an equal probability of not covering the parameter at each end of the interval.

DERIVATION OF THE INTERVALS

The usual method of deriving confidence limits for a set of parameters is to find functions of these parameters and their estimators that have distributions independent of the parameters. Such functions are called pivotal quantities. The distributions of these pivotal quantities can be tabulated and by using the inverse of the original function confidence limits can be calculated. The problems associated with this method are firstly finding the pivotal quantities and secondly finding their distributions. In the case of the FT 1 pivotal quantities are easily found, for A and B $(A-\hat{A})/\hat{B}$ and $B/\hat{B}$ are pivotal (for further details see Lawless (1978)), however their distributions are intractable.

Lawless (1972, 1973b) derives the conditional distributions of these pivotal quantities given certain functions of the sample, called ancillary statistics. These ancillary statistics are similar to pivotal quantities in that they have distributions that do not involve the parameters. In this case they take the form $(x-\hat{A})/\hat{B}$. While these conditional distributions are of interest, and some statisticians would maintain that they should be used in preference to the unconditional ones (Lawless (1973a)), they have to be recalculated for every sample and so are inconvenient for routine application.

Although the unconditional distributions cannot be obtained analytically some percentage points have been produced by Monte Carlo methods (e.g. Thoman et al (1969)). This method is very expensive of computer time and so only a limited number of sample sizes and percentage points have been calculated.

Lawless (1974) proposes approximations to the unconditional distributions by taking expectations of the ancillary statistics in the unconditional distributions. When compared with the results of Thoman et al this approximation gives very good agreement. It is these approximate distributions given in Lawless (1974) that have been used to calculate the percentage points given here. It should be noted that throughout Lawless works with the distribution of minima, - X in our notation.

PRESENTATION

The percentage points for the various statistics were calculated for ample sizes 5, 6, 7, 10, 12, 15, 20, 30, 40 and 60. These are given in tables below. In addition to these tables various graphs showing 90, 95 and 99% confidence intervals for increasing sample size have been drawn. In the following description of how to use these tables and graphs, z_1 and z_2 will be the lower and upper values respectively from the graph or table and $\hat{A}$ and $\hat{B}$ the maximum likelihood estimates of A and B.

LOCATION PARAMETER A

Table 1 and Figure 1 give the percentage points needed to calculate 90%, 95% and 99% confidence intervals for the location parameter A. Once z_1 and z_2 have been obtained the intervals are calculated as

$$\hat{A}+z_1\hat{B} < \hat{A}+z_2\hat{B}$$

As an example consider the January extreme wave height data in metres from the Seven Stones Light Vessel (for details see Carter and Challenor (1978)). Here there are 7 years data and the maximum likelihood estimates of A and B are 8.08m and 0.4m respectively. From table 1 z_1 and z_2 for a 90% interval with sample size 7 are -0.829 and 0.874. Therefore the 90% confidence interval is

$$7.75\text{m} < A < 8.43\text{m}$$

SCALE PARAMETER B

Table 2 (and Figure 2) gives the percentage points necessary to produce confidence intervals for B, the scale parameter of the FT 1 distribution. The intervals are given by

$$\hat{B}/z_2 < B < \hat{B}/z_1$$

Therefore using our previous example, z_1 and z_2 are 0.458 and 1.410 respectively which gives a 90% confidence interval

$$0.28\text{m} < B < 0.87\text{m}$$

PERCENTILES AND RETURN VALUES

Tables 3, 4 and 5 (and the corresponding Figures) give the percentage points to calculate confidence intervals for the median (the 50th percentile), the 98th percentile and the 99th percentile respectively. If the observations are annual then these are the 2, 50 and 100 year return values. The confidence intervals are given by

$$\hat{A}+z_1\hat{B} < y_p < \hat{A}+z_2\hat{B}$$

where y_p is the pth percentile.

(N.B. The signs have all been reversed in the formula in Carter and Challenor (1978)).

Using the same example as before, z_1 and z_2 are 2.513 and 8.706 respectively hence the 90% confidence interval is

$$9.09\text{m} < y_{98} \text{ (}=H_{50} \text{ in this case)} < 11.56\text{m}$$

PREDICTION LIMITS

Prediction limits differ from confidence limits in that we no longer have a parameter to estimate but some future value of a random variable to predict. Even if we knew the parameters we could wtill not predict a future observation exactly. These minimum intervals are given in the tables under sample size∞.

Tables 6, 7, 8 and 9 give the percentage points to calculate prediction limits for the largest in samples size 2, 10, 50 and 100 respectively. The limits are given by

$$\hat{A}+z_1\hat{B} < Y_{(m)} < A+z_2\hat{B}$$

where $Y_{(m)}$ is the largest in a sample size m.

Taking our example for the last time, if we are interested in samples size 50, z_1 and z_2 are 2.38 and 11.05 so the 90% prediction limits are

$$9.03\text{m} < Y_{(50)} < 12.50\text{m}$$

CONCLUSIONS

The results presented in this note enable confidence limits to be produced for the usual statistics calculated from extreme data. They are also useful in deciding how long sampling should continue by giving an estimate of the precision obtained at each stage.

However, it should be noted that these limits have been derived assuming the data is sampled from a FT 1 distribution.

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P(Z< z)

Sample size	0.005	0.025	0.05	0.95	.975	0.995
5	-2.311	-1.416	-1.107	1.247	1.638	2.835
6	-1.850	-1.190	-0.939	1.007	1.315	2.130
7	-1.580	-1.044	-0.829	0.874	1.120	1.740
10	-1.152	-0.798	-0.644	0.665	0.820	1.190
12	-1.000	-0.704	-0.572	0.587	0.714	1.015
15	-0.855	-0.610	-0.499	0.509	0.612	0.850
20	-0.702	-0.511	-0.421	0.428	0.509	0.693
30	-0.551	-0.404	-0.334	0.338	0.400	0.536
40	-0.465	-0.344	-0.285	0.288	0.341	0.453
60	-0.371	-0.277	-0.229	0.230	0.274	0.361

Table 1 : Percentage Points for $Z = (A-\hat{A})/\hat{B}$

P(Z<z)

Sample size	0.005	0.025	0.05	0.95	0.975	0.995
5	0.190	0.293	0.360	1.464	1.560	1.856
6	0.246	0.352	0.411	1.435	1.533	1.789
7	0.292	0.398	0.458	1.410	1.502	1.736
10	0.394	0.496	0.553	1.355	1.434	1.623
12	0.443	0.541	0.595	1.330	1.401	1.571
15	0.498	0.590	0.639	1.299	1.364	1.513
20	0.562	0.646	0.690	1.264	1.320	1.446
30	0.639	0.712	0.750	1.220	1.265	1.365
40	0.687	0.751	0.786	1.191	1.231	1.317
60	0.743	0.798	0.824	1.159	1.191	1.259

Table 2 : Percentage Points for $Z = \hat{B}/B$

Sample size	P(Z<z) 0.005	0.025	0.05	0.95	0.975	0.995
5	-1.837	-0.972	-0.676	1.848	2.315	3.630
6	-1.356	-0.750	-0.524	1.606	1.965	2.910
7	-1.083	-0.612	-0.426	1.448	1.744	2.490
10	-0.687	-0.389	-0.258	1.182	1.385	1.860
12	-0.554	-0.307	-0.193	1.083	1.254	1.646
15	-0.427	-0.223	-0.126	0.982	1.124	1.440
20	-0.300	-0.136	-0.055	0.878	0.992	1.239
30	-0.167	-0.039	0.025	0.767	0.853	1.034
40	-0.092	0.016	0.071	0.706	0.777	0.925
60	-0.007	0.080	0.125	0.637	0.692	0.806

Table 3 : Percentage Points for $Z = (y_{50} - \hat{A})/\hat{B}$

P(Z<z)

Sample size	0.005	0.025	0.05	0.95	0.975	0.995
5	1.581	2.075	2.372	11.148	13.557	20.780
6	1.705	2.172	2.448	9.634	11.372	16.245
7	1.808	2.252	2.513	8.706	10.075	13.740
10	2.035	2.432	2.662	7.268	8.128	10.257
12	2.146	2.521	2.736	6.766	7.469	9.155
15	2.276	2.652	2.823	6.284	6.846	8.152
20	2.435	2.753	2.930	5.813	6.246	7.220
30	2.642	2.918	3.070	5.337	5.649	6.328
40	2.775	3.024	3.160	5.088	5.339	5.878
60	2.945	3.158	3.274	4.821	5.011	5.411

Table 4 : Percentage Points for $Z = (y_{98} - \hat{A})/\hat{B}$

$P(Z<z)$

Sample size	0.005	0.025	0.05	0.95	0.975	0.995
5	1.983	2.540	2.877	13.084	15.910	24.401
6	2.118	2.645	2.959	11.297	13.331	19.040
7	2.230	2.732	3.030	10.205	11.804	16.100
10	2.481	2.932	3.193	8.516	9.518	12.000
12	2.605	3.030	3.275	7.928	8.746	10.710
15	2.750	3.147	3.373	7.365	8.017	9.535
20	2.929	3.290	3.493	6.815	7.316	8.448
30	3.161	3.476	3.650	6.261	6.621	7.409
40	3.313	3.597	3.752	5.971	6.262	6.885
60	3.505	3.750	3.882	5.661	5.881	6.344

Table 5 : Percentage Points for $Z = (y_{99} - \hat{A})/\hat{B}$

	$P(Z < z)$					
Sample size	0.005	0.025	0.05	0.95	0.975	0.995
5	-2.48	-1.30	-0.86	6.06	7.83	13.00
6	-2.05	-1.13	-0.76	5.49	6.96	11.00
7	-1.81	-1.03	-0.70	5.14	6.44	9.88
10	-1.47	-0.88	-0.60	4.60	5.66	8.31
12	-1.37	-0.83	-0.56	4.42	5.40	7.82
15	-1.27	-0.78	-0.52	4.25	5.16	7.38
20	-1.19	-0.74	-0.50	4.09	4.94	6.98
30	-1.11	-0.69	-0.47	3.94	4.74	6.62
40	-1.07	-0.67	-0.44	3.87	4.64	6.45
60	-1.04	-0.65	-0.43	3.80	4.55	6.29
∞	-0.97	-0.61	-0.40	3.66	4.37	5.99

Table 6 : Percentage Points for $Z = (Y_{(2)} - \hat{A})/\hat{B}$

P(Z<z)

Sample size	0.005	0.025	0.05	0.95	0.975	0.995
5	0.05	0.60	0.90	9.72	12.13	19.27
6	0.15	0.66	0.95	8.63	10.52	15.76
7	0.21	0.70	0.98	7.97	9.57	13.84
10	0.33	0.78	1.04	6.96	8.18	11.23
12	0.38	0.81	1.06	6.63	7.73	10.43
15	0.43	0.84	1.08	6.31	7.32	9.73
20	0.47	0.88	1.11	6.03	6.94	9.10
30	0.53	0.92	1.14	5.76	6.59	8.55
40	0.55	0.93	1.15	5.63	6.43	8.29
60	0.58	0.95	1.17	5.51	6.27	8.05
∞	0.64	1.00	1.21	5.27	5.98	7.60

Table 7 : Percentage Points for $Z = (Y_{(10)} - \hat{A})/\hat{B}$

P(Z<z)

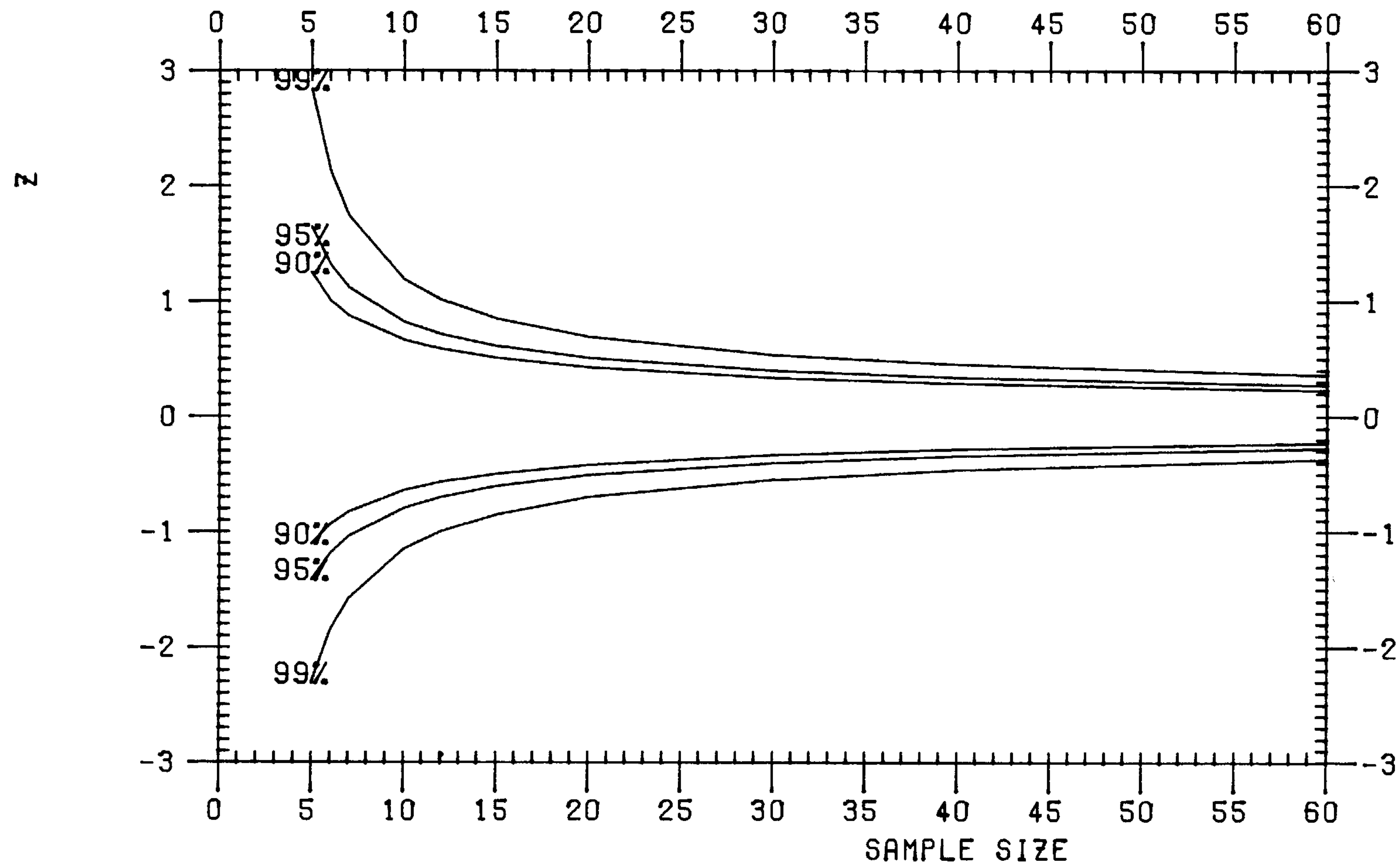
Sample size	0.005	0.025	0.05	0.95	0.975	0.995
5	1.33	1.92	2.29	13.78	16.97	26.41
6	1.42	1.99	2.33	12.08	14.48	21.21
7	1.49	2.04	2.38	11.05	13.03	18.34
10	1.65	2.16	2.47	9.50	10.92	14.46
12	1.72	2.21	2.50	8.98	10.22	13.28
15	1.79	2.27	2.55	8.50	9.60	12.25
20	1.88	2.33	2.60	8.05	9.03	11.34
30	1.98	2.41	2.66	7.63	8.50	10.54
40	2.03	2.45	2.69	7.43	8.26	10.17
60	2.09	2.50	2.73	7.25	8.02	9.83
∞	2.24	2.61	2.81	6.88	7.59	9.21

Table 8 : Percentage Points for $Z = (Y_{(50)} - \hat{A})/\hat{B}$

P(Z<z)

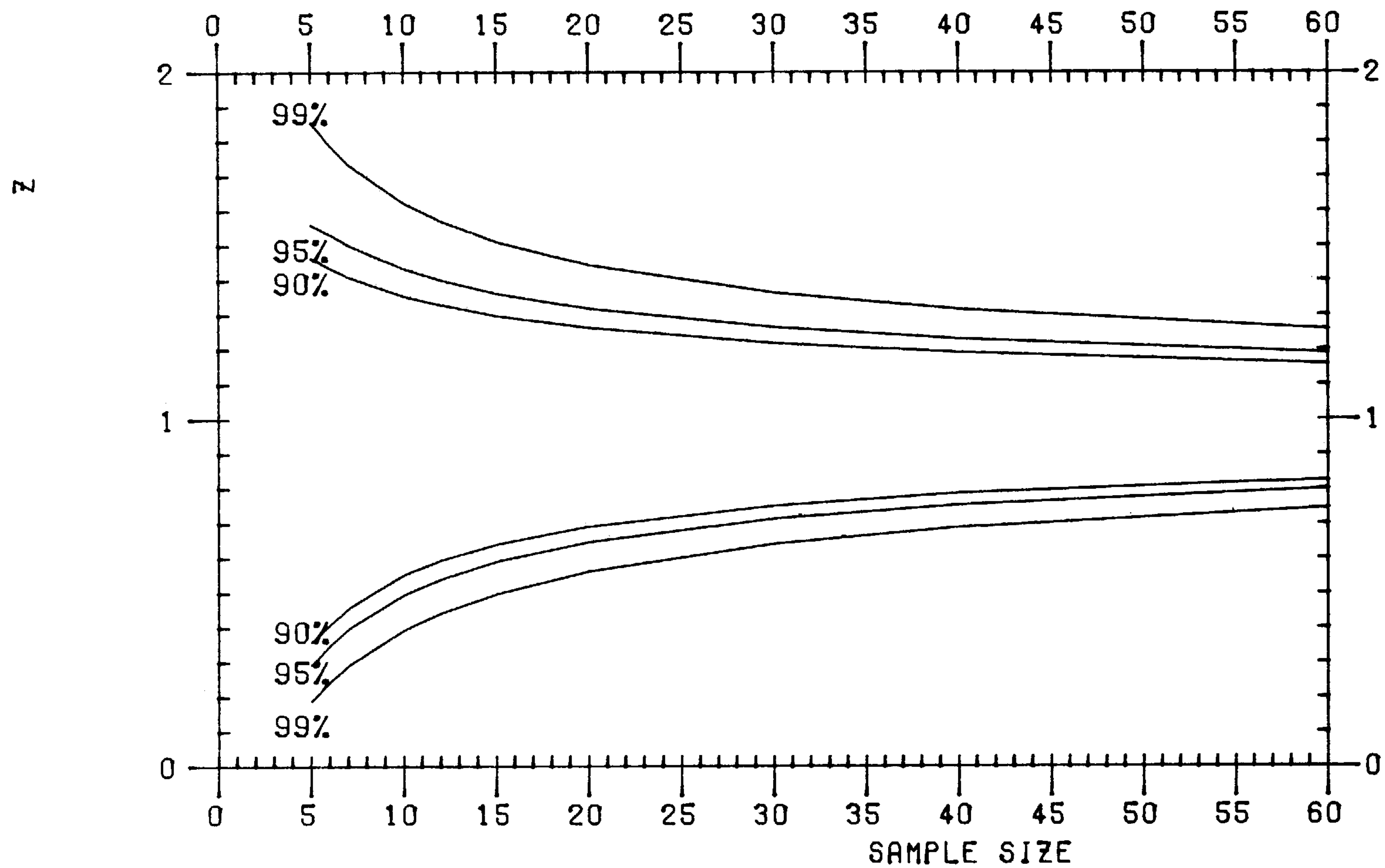
Sample size	0.005	0.025	0.05	0.95	0.975	0.995
5	1.79	2.44	2.84	15.59	19.15	29.58
6	1.89	2.52	2.89	13.62	16.27	23.72
7	1.98	2.58	2.94	12.43	14.60	20.40
10	2.16	2.71	3.04	10.63	12.14	15.93
12	2.25	2.78	3.10	10.03	11.35	14.57
15	2.33	2.85	3.15	9.46	10.62	13.38
20	2.44	2.93	3.22	8.94	9.95	12.34
30	2.57	3.03	3.29	8.45	9.35	11.42
40	2.65	3.09	3.34	8.22	9.06	11.00
60	2.73	3.15	3.39	8.00	8.79	10.61
∞	2.94	3.30	3.51	7.58	8.28	9.90

Table 9 : Percentage Points for $Z = (Y_{(100)} - \hat{A})/\hat{B}$



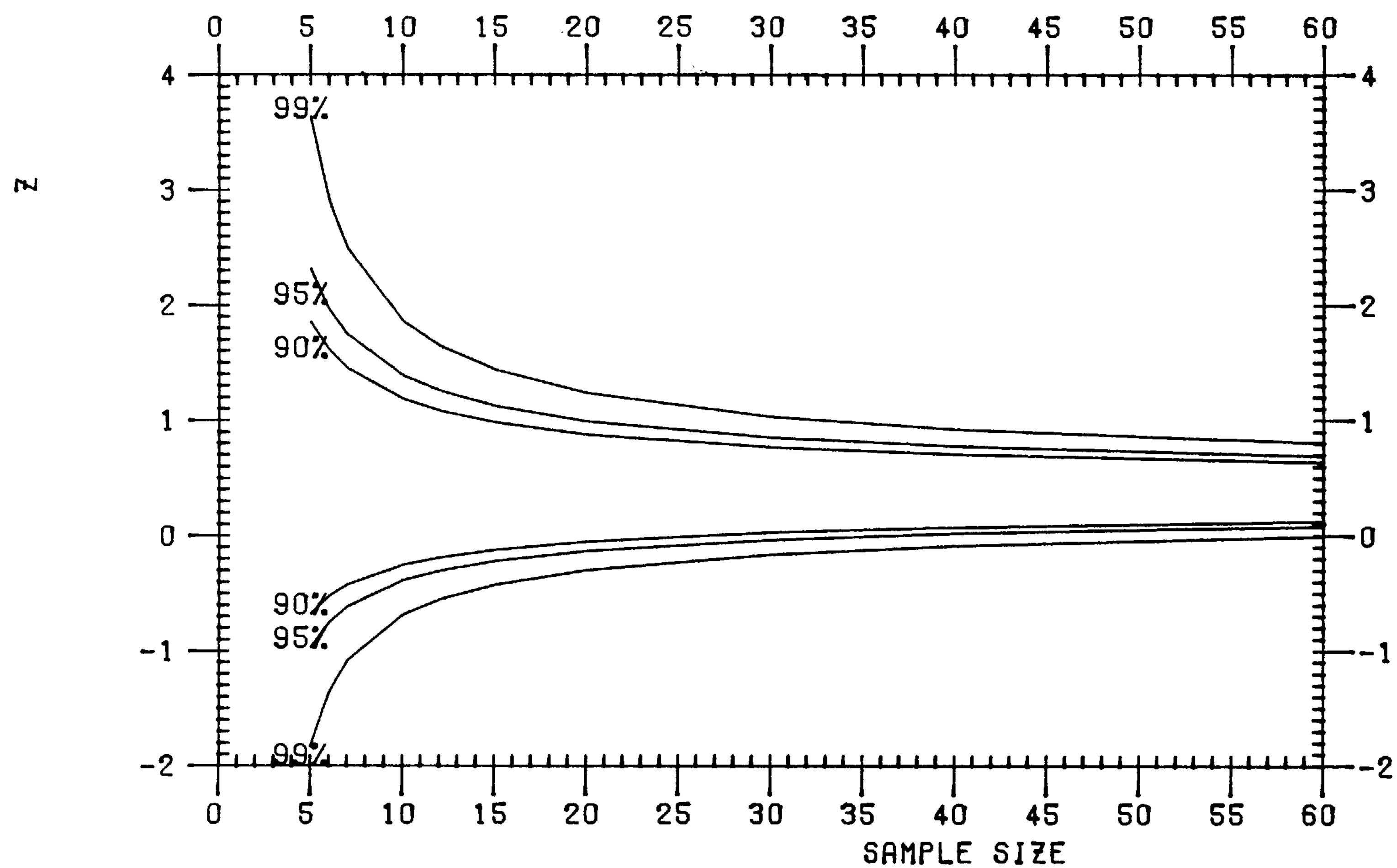
Confidence Limits for $Z = (A - \hat{A})/\hat{B}$

Fig 1



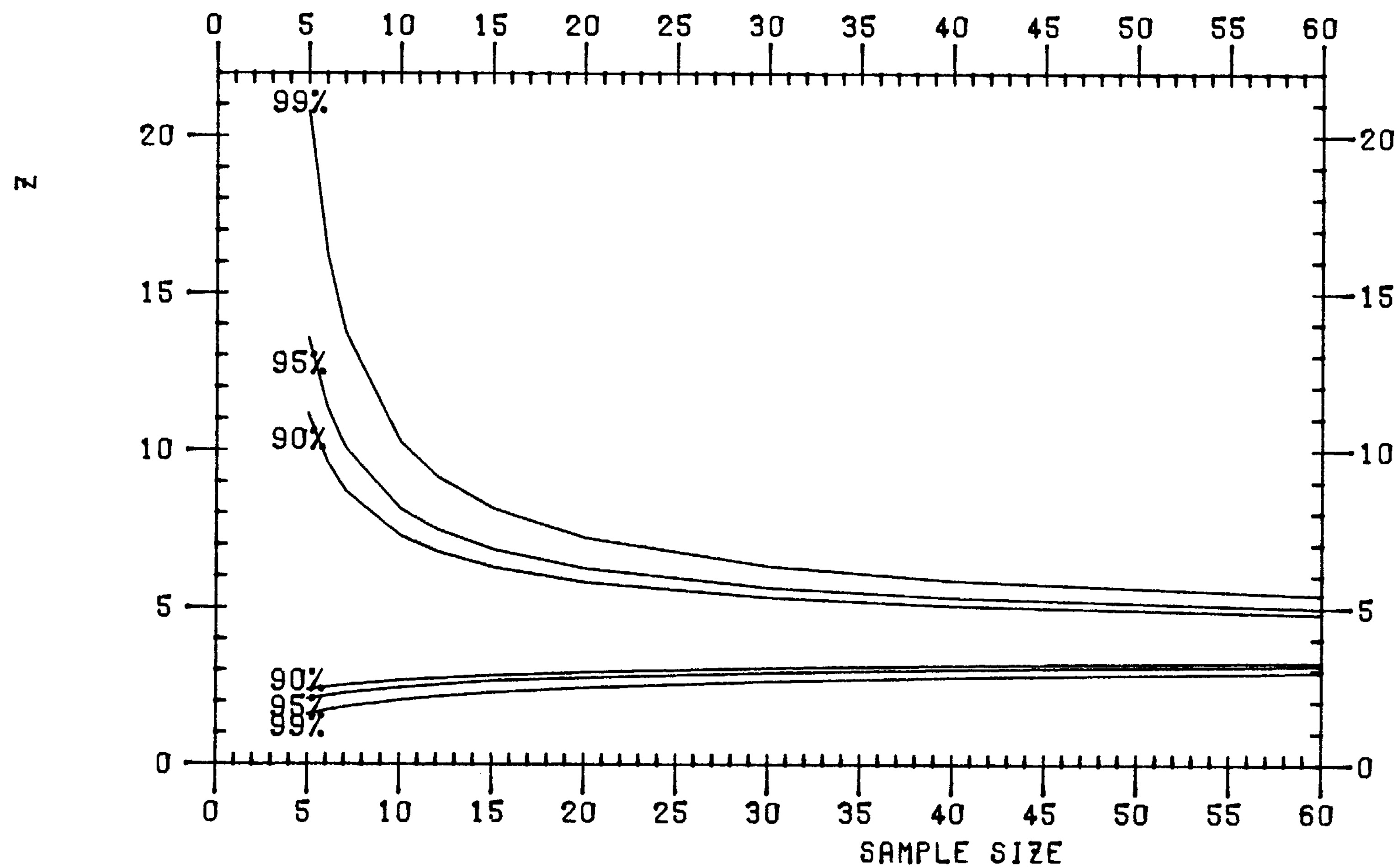
Confidence Limits for $Z = \hat{B}/B$

Fig 2



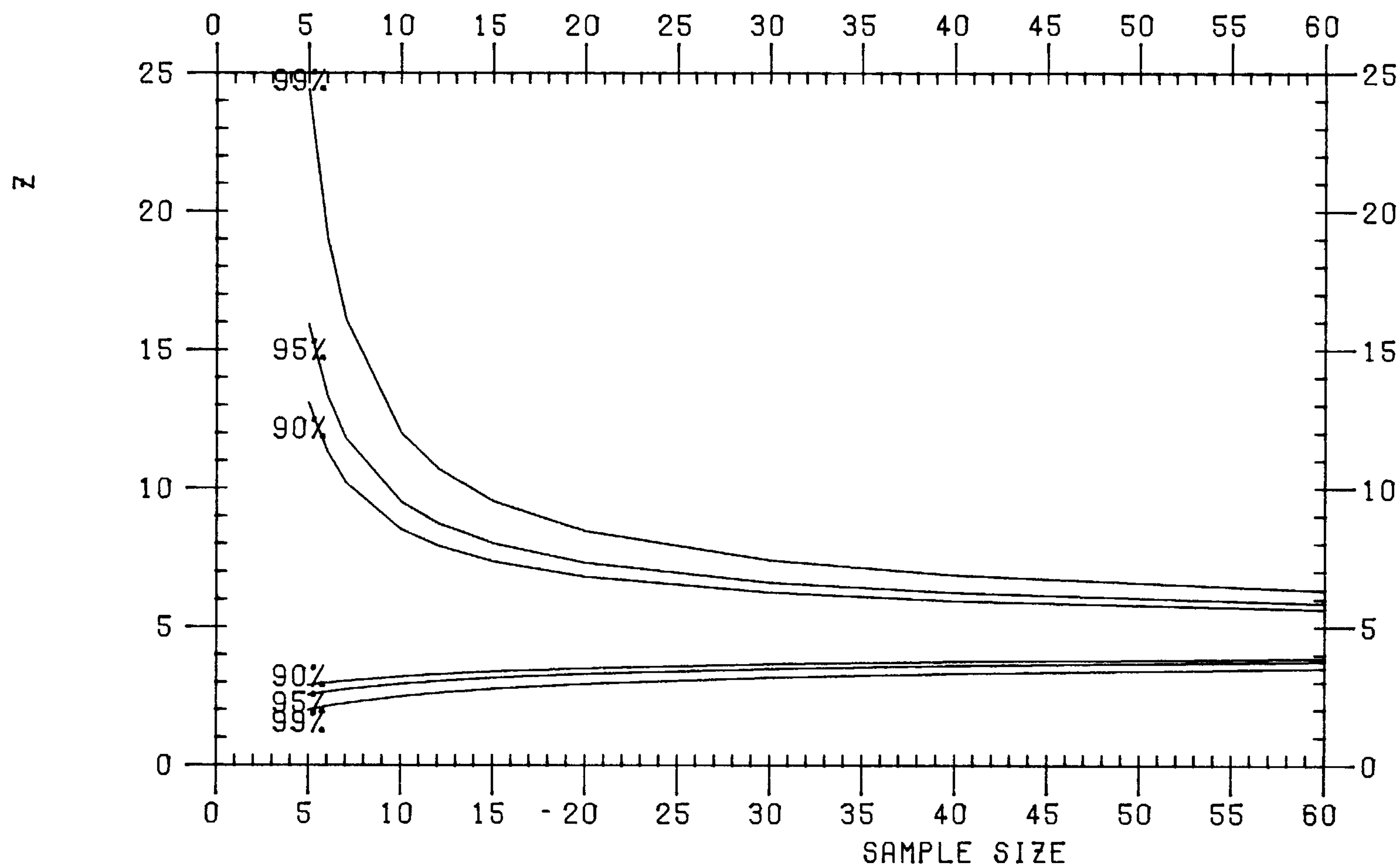
Confidence Limits for $Z = (y_{50} - \hat{A}) / \hat{B}$

Fig 3



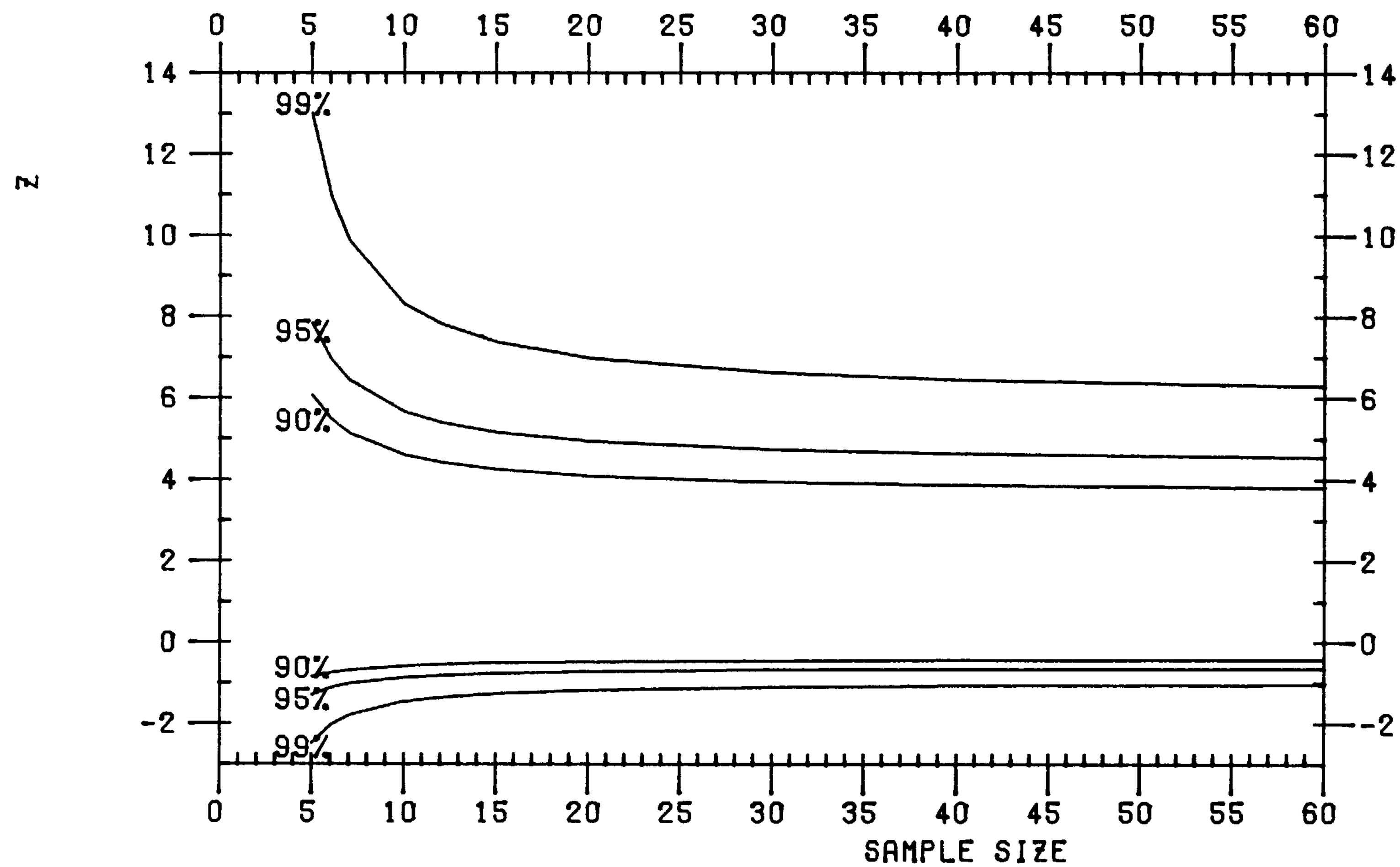
Confidence Limits for $Z = (y_{98} - \hat{A})/\hat{B}$

Fig4



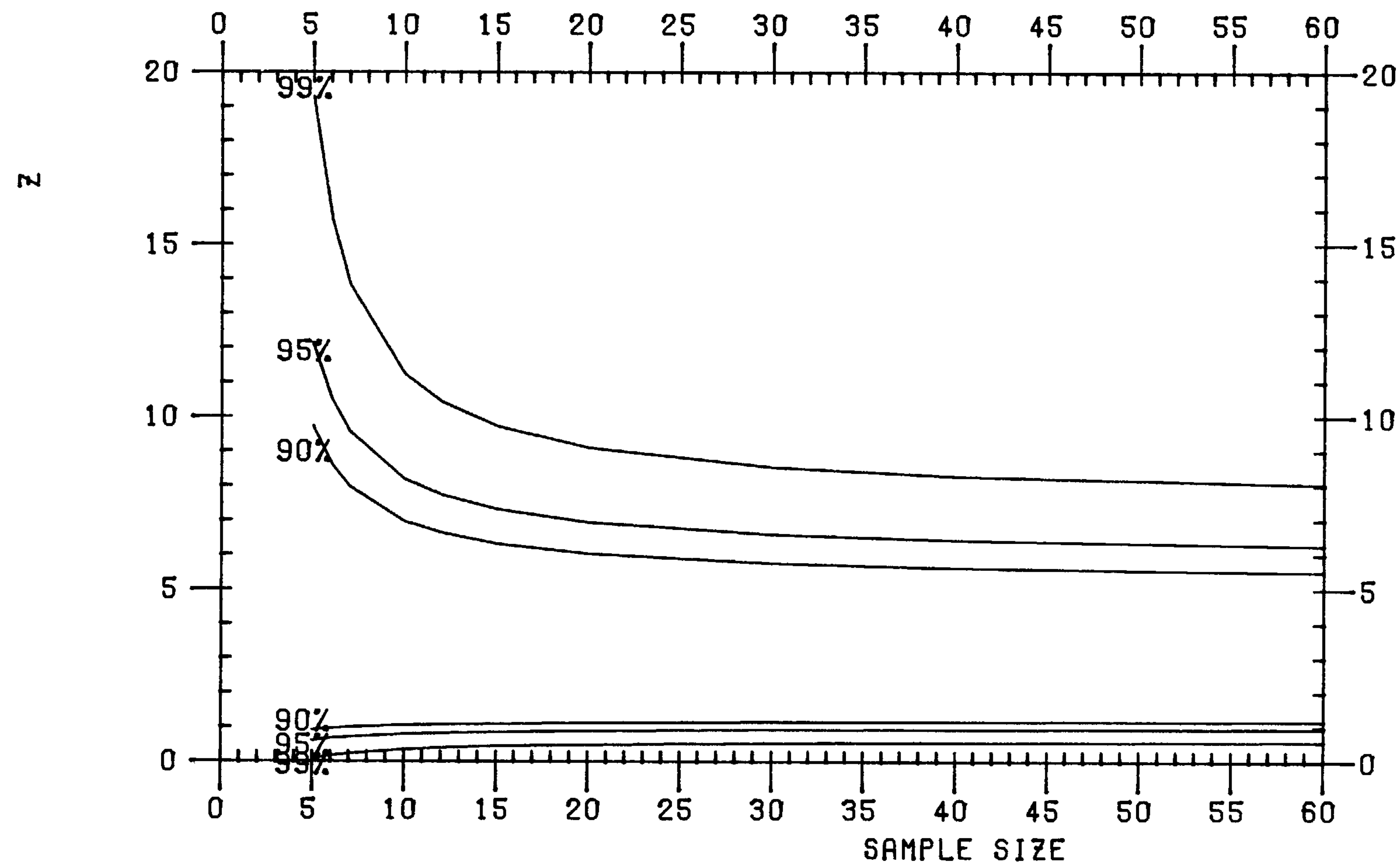
Confidence Limits for $Z=(y_{99}-\hat{A})/\hat{B}$

Fig 5



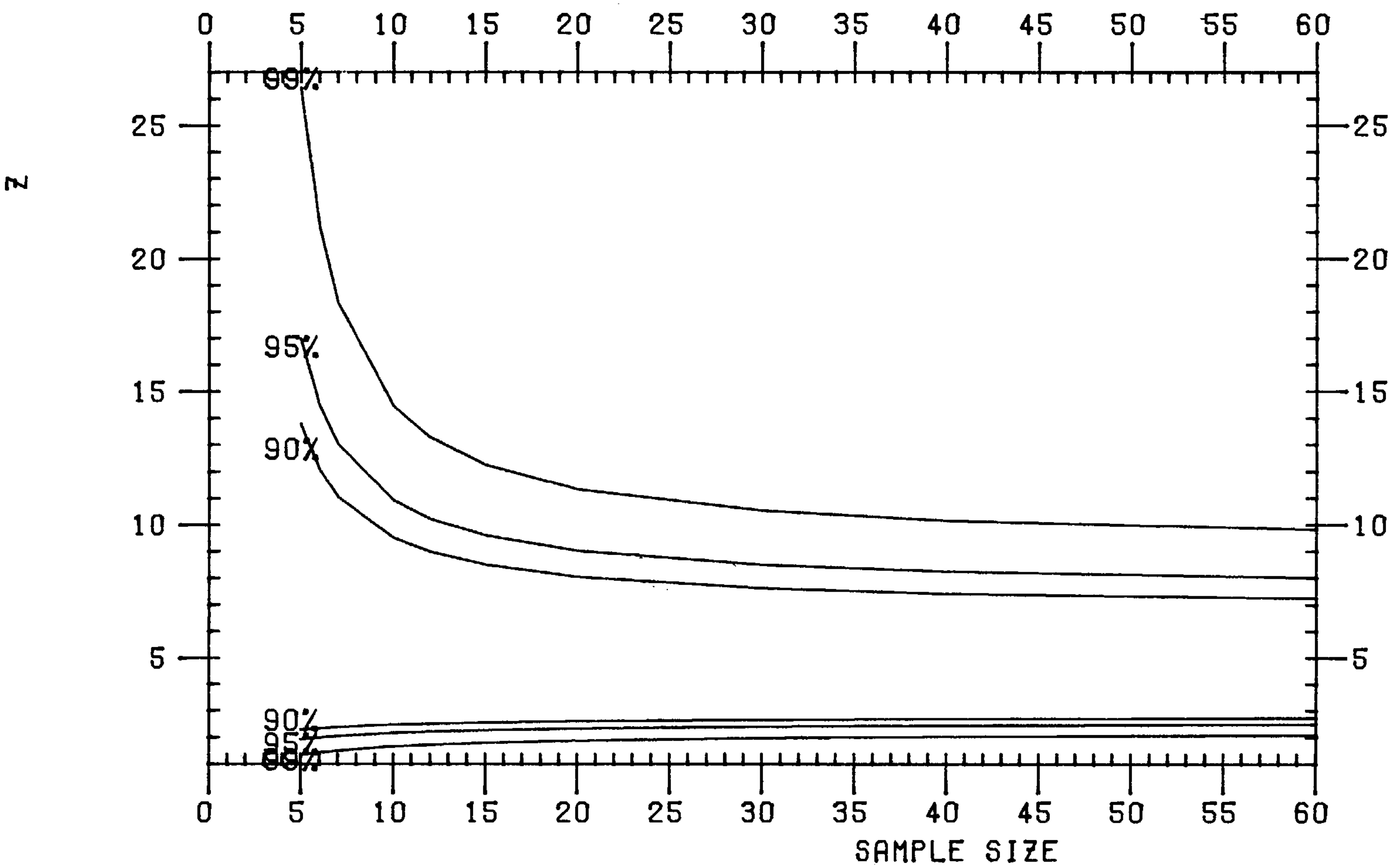
Prediction Limits for $Z=(Y_{(2)}-\hat{A})/\hat{B}$

Fig 6



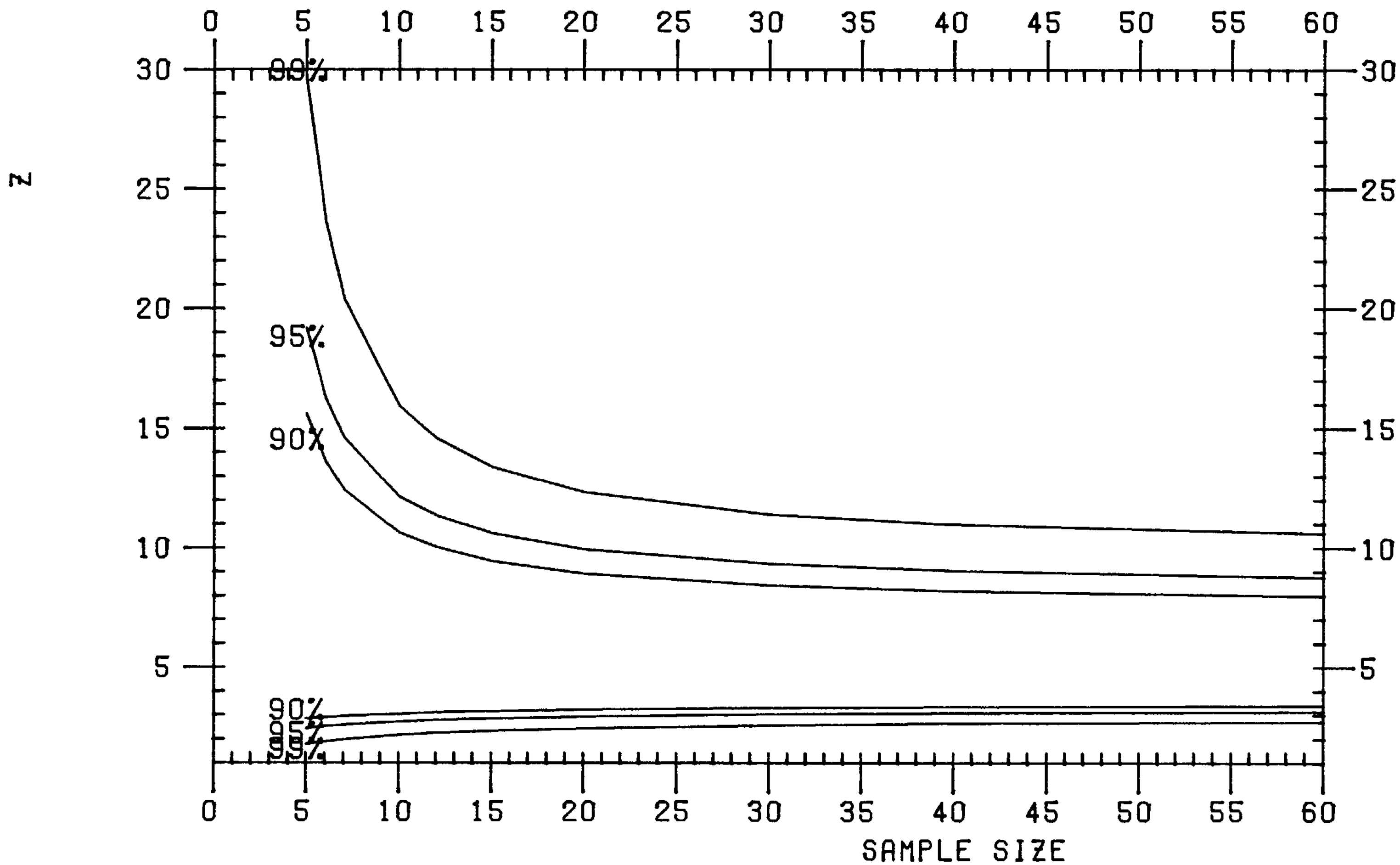
Prediction Limits for $Z=(Y_{(10)}-\hat{A})/\hat{B}$

Fig 7



Prediction Limits for $Z=(Y_{(50)}-\hat{A})/\hat{B}$

Fig8



Prediction Limits for $Z=(Y_{(100)}-\hat{A})/\hat{B}$

Fig 9